First-order Modal Logic

A brief introduction

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Background

First-order Modal Logic

First-order Logic:

- (First-order) Quantifiers $\forall, \exists$
- Predicates to form atomic formulas $P(\cdot), R(\cdot, \cdot), \approx$ ...
- Terms: variables, constants, function symbols

(Propositional) Modal Logic:

- Propositional Language
- Modalities $\Box, \Diamond, \mathcal{K}, \mathcal{G}, [\pi] \ldots$

Alternative names: Quantified Modal Logic, Modal Predicate Logic

Mathematical vs. Philosophical Logic

Capture mathematical or philosophical theories by axioms on top of the background logic.

Mathematical theory	Philosophical theory
First-order/Second-order logics	Modal/Non-classical logics
Mostly syntactically driven	Syn./semantically driven
$\forall x \forall y \forall z \, (x \cdot y) \cdot z = x \cdot (y \cdot z)$	$\mathcal{K}p \to \mathcal{K}\mathcal{K}p$
$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \land z \in w))$	$\Box(\Box p \to p) \to \Box p$

In philosophical logic, (linguistic) intuition about the concepts in natural language plays an important role.

The interactions between quantifiers and modalities are even more interesting, which dates back to the origin of Logic.

Aristotle's modal syllogisms

Chapters 3 and 8-22 of the first book of the *Prior Analytics*.

A natural extension of the syllogistic language:

- Quantity: all, some
- Quality: affirmative (is), negative (not)
- *Modes*: null (X), necessary (L), possible (M), contingent (C)
- 16 types of sentences

Aristotle's modal syllogistic

Syllogisms: All A is B (A**a**B), No A is B (A**e**B), Some A is B (A**i**B), Some A is not B (A**o**B): e.g., Celarent, Darii, Ferio.

For example: B**a**rb**a**r**a** LXL is considered valid by Aristotle:

$$\frac{\text{All } B \text{ is necessarily } A \qquad \text{All } C \text{ is } B}{\text{All } C \text{ is necessarily } A}$$

How to read: All *B* is necessarily *A* (the B-things are necessarily *A*) vs. Necessarily All *B* is *A* (the statement is necessary).

***De re* reading**

$$\frac{\forall x(B(x) \to \Box A(x)) \quad \forall x(C(x) \to B(x))}{\forall x(C(x) \to \Box A(x))}$$

De dicto reading

$$\frac{\Box\forall x(B(x) \to A(x)) \quad \forall x(C(x) \to B(x))}{\Box\forall x(C(x) \to A(x))}$$

de re vs. *de dicto*

From the Barbaras, it seems Aristotle advocates the *de re* reading.

However, his acceptance of some other conversion rules suggests *de dicto* reading:

- Necessarily Some B is A $\Box(\exists x(B(x) \wedge A(x)))$
- therefore: Necessarily Some A is B $\Box(\exists x(A(x) \wedge B(x)))$

Compare it with the intuitively invalid *de re* counterpart:

- Some B is necessarily A $\exists x(B(x) \wedge \Box A(x))$
- therefore: Some A is necessarily B $\exists x(A(x) \wedge \Box B(x))$

Making sense?

"Aristotle's modal syllogistic is almost incomprehensible because of its many faults and inconsistencies".

–Łukasiewicz

There is a body of work trying to propose other interpretations or impose further philosophical conditions to make sense of Aristotle's original modal syllogistic, e.g., [Rini 2011], [Malink 2013] as some recent examples.

Without a formal language to disambiguate the possible readings and a rigorous semantics, it is hard to argue for the meaning of sentences with both quantifiers and modalities.

Beyond historical interest

Modal syllogisms are *natural* and *abundant*.

In the epistemic setting:

- Some flight from Delhi is to Kolkata,
- All flights to Kolkata are known to have been canceled,
- therefore: Some flight from Delhi is known to have been canceled.

In the deontic setting:

- All college students are adults,
- All adults are permitted to drink beers,
- therefore: All college students are permitted to drink beers.

Beyond syllogisms

The interactions of quantifiers and modalities are important in modern applications of logic.

- Capturing metaphysical theories: All humans are necessarily mortal ($\forall\Box$); there could have been a unicorn ($\Diamond\exists$); Necessarily everything is necessarily something ($\Box\forall\Box\exists$)...
- Capturing *de re* knowledge in philosophy, TCS and AI, such as know-how/why/who and so on. E.g., *knowing how to prove a given theorem* can be formalized by *there is a method such that you know that it can prove the theorem* ($\exists\Box$ vs. $\Box\exists$).
- Capturing (temporal) properties of computer systems, e.g., every process must eventually terminate ($\forall\Diamond$ pattern).
- Deontic reasoning, multi-agent epistemic reasoning, multi-verse mathematics...

Early days of First-order Modal Logic

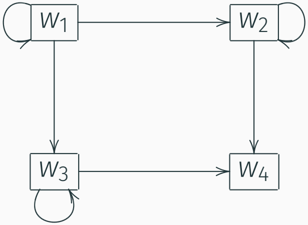
Ruth Barcan Marcus (1946) explored first-order S2 and highlighted the following important formuals:

$$\begin{array}{rll} \text{Barcan:} & \forall x\Box\varphi \to \Box\forall x\varphi & \text{equiv. } \Diamond\exists\varphi \to \exists x\Diamond\varphi \\ \text{Converse Barcan:} & \Box\forall x\varphi \to \forall x\Box\varphi & \text{equiv. } \exists x\Diamond\varphi \to \Diamond\exists\varphi. \end{array}$$

She also discovers the results about the necessity of identity and deduction theorems.

Saul Kripke (1959) proves the completeness of first-order S5 logic based on the possible world semantics.

Kripke model for FOML



Instead of the valuations in the propositional modal logic, each possible world w_i is a first-order structure (D, I), where D is a non-empty domain, and I gives interpretations to non-logcial symbols such as the predicate symbols.

Optimistic early view on FOML

In *Meaning and Necessity* (1947), Carnap remarked:

> *Any system of modal logic without quantification is of interest only as a basis for a wider system including quantification. If such a wider system were found to be impossible, logicians would probably abandon modal logic entirely.*

It was indeed the case in early textbooks on modal logic, e.g., Hughes and Cresswell (1968): PML is the preparation for FOML.

However, it seems that history went exactly the other way around.

First-order modal logic is infamous for

- issues and choices in the models and the semantics
- *quantifying-in* and substitution
- ambiguity: *de re* vs. *de dicto*
- incompleteness
- lack of Craig's interpolation
- undecidablility (hard to find useful decidable fragments)
- ....

It is definitely not a simple combination of FOL and PML.

At the same time

Propositional modal logic has been too successful:

- natural language, simple semantics, neat logics, and computationally good in general
- yet technically non-trivial to work on to obtain deep results
- applications in philosophy, TCS, and AI...
- connections with non-classical (propositional) logics

At the same time

In 1200-page *Handbook of Modal Logic* (2007), FOML only occupies one out of 21 chapters. In textbook *Modal logic* (2001) by Blackburn et al. there is nothing about first-order modal logic.

In *Handbook of Epistemic Logic* (2015), there is hardly anything about quantified epistemic logic that is crucial for *de re* knowledge.

FOML was not the focus, though scattered work continues to advance the field, e.g., Fitting and Mendelsohn (1998).

Time to go back to FOML

The past decade witnessed the resurgence of FOML:

- Many conceptual issues are clarified
- Basic results have been proven
- New frameworks have been invented
- New techniques have been developed
- New well-behaved fragments have been discovered
- New applications have been found
- Courses, workshops, seminars on FOML
- Research in propositional modal logic is getting saturated....

The later lectures will give you a taste of the new developments, but let us start low with the basics.

Kripke Models with Domain Conditions

Language of first-order modal logic

A signature Σ of non-logical symbols includes: for each natural number $n \in \mathbb{N}$, a set Rel^n of n-ary predicate symbols, and a set of variables Var. Assume these sets are at most countable. For simplicity, we leave out the function symbols (and constant symbols). Therefore the terms are only variables.

Definition (Language $\mathcal{L}^{\Sigma}_{\approx}$)

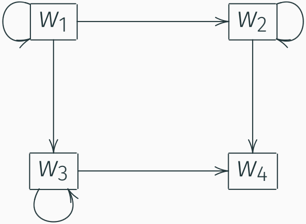
Given a signature Σ, $\mathcal{L}^{\Sigma}_{\approx}$-formulas are defined as:

$$\varphi ::= P^n(x_1 \cdots x_n) \mid (x \approx y) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \forall x\varphi$$

where $x, y, x_i \in Var, P^n \in Rel^n$. $\vee, \to, \Diamond, \exists$ are abbreviations.

Free and bound occurrences of variables are defined as in the FOL case, so is the admissibility of substitutions. We use $\varphi[y/x]$ to denote the result of admissible substitution of x by y in φ.

Recall the intuition of Kripke model for FOML



Instead of the valuations in the propositional modal logic, each possible world w_i is a first-order structure (D, I), where D is a non-empty domain, and I gives interpretations to non-logical symbols such as the predicate symbols.

The domains of the first-order structures on different worlds are not necessarily the same, but let us start with the simplest setting.

Kripke models with a constant domain

Definition (Constant Domain Model)

A constant domain model is a tuple $\mathcal{M} = \langle W, R, D, I \rangle$ where:

- W is a non-empty set (possible worlds);
- $R \subseteq W \times W$ (accessibility relation);
- D is a non-empty domain;
- For each $w \in W$, $I(w)$ is a function assigning a n-ary relation to P^n, i.e., $I(w)(P^n) \subseteq D^n$.

$\langle W, R, D \rangle$ is called a skeleton and $\langle W, R \rangle$ is called a frame.

Assignments

To interpret formulas with free variables, we also need the assignment for variables.

Definition (Assignment)

Given a model $\mathcal{M}$, an assignment σ is a function assigning each variable an element in the domain, i.e., $\sigma : Var \to D$. For $a \in D$, $\sigma(a/x)$ is the modified assignment such that:

- $\sigma(a/x)(x) = a$
- $\sigma(a/x)(y) = \sigma(y)$ for all variables $y \neq x$

If we were to introduce constants in the language, they are non-rigid (world-dependent) in general, like the interpretation of predicates. In contrast, the variables are rigid due to assignments.

Semantics

Definition (Truth Conditions)

Given a model $\mathcal{M}$, world w in it, and an assignment σ:

$$
\begin{aligned}
\mathcal{M}, w, \sigma \models P^n(x_1 \cdots x_n) &\quad\Leftrightarrow\quad \langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in I(w)(P^n) \\
\mathcal{M}, w, \sigma \models x \approx y &\quad\Leftrightarrow\quad \sigma(x) = \sigma(y) \\
\mathcal{M}, w, \sigma \models \neg\varphi &\quad\Leftrightarrow\quad \mathcal{M}, w, \sigma \not\models \varphi \\
\mathcal{M}, w, \sigma \models (\varphi \wedge \psi) &\quad\Leftrightarrow\quad \mathcal{M}, w, \sigma \models \varphi \text{ and } \mathcal{M}, w, \sigma \models \psi \\
\mathcal{M}, w, \sigma \models \Box\varphi &\quad\Leftrightarrow\quad \text{for all } v \text{, if } wRv \text{ then } \mathcal{M}, v, \sigma \models \varphi \\
\mathcal{M}, w, \sigma \models \forall x\varphi &\quad\Leftrightarrow\quad \text{for all } a \in D,\ \mathcal{M}, w, \sigma(a/x) \models \varphi
\end{aligned}
$$

φ is valid in a model $\mathcal{M}$ if for each worlds w each assignment σ, $\mathcal{M}, w, \sigma \models \varphi$. φ is valid in a frame (skeleton) if it is valid in all constant-domain models based on the frame (skeleton).

Both (BF) $\forall x \Box\varphi \to \Box\forall x\varphi$ and (CBF) $\Box\forall x\varphi \to \forall x\Box\varphi$ are both valid over any constant-domain models.

Recall: Aristotle's modal syllogistic

Barbara LXL is considered valid by Aristotle:

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How to read: All B is necessarily A (the B-things are necessarily A) vs. Necessarily All B is A (the statement is necessary).

De re reading
$$\frac{\forall x(B(x) \to \Box A(x)) \quad \forall x(C(x) \to B(x))}{\forall x(C(x) \to \Box A(x))}$$

De dicto reading
$$\frac{\Box\forall x(B(x) \to A(x)) \quad \forall x(C(x) \to B(x))}{\Box\forall x(C(x) \to A(x))}$$

System QS: merging FOL and PML

Definition (Axioms of QS)

Given a normal propositional modal logic S, QS is defined as:

- S All FOML substitution instances of S-theorems
- I1 $x \approx x$
- I2 $x \approx y \to (\varphi[x/z] \to \psi[y/z])$
- $\Box$NI $x \not\approx y \to \Box(x \not\approx y)$ (not for terms with constants)
- $\forall$E $\forall x \varphi \to \varphi[y/x]$

Rules:

$$\frac{\varphi \to \psi \quad \varphi}{\psi}\ \text{MP} \qquad \frac{\vdash \varphi}{\vdash \Box\varphi}\ \text{N} \qquad \frac{\vdash \varphi \to \psi}{\vdash \varphi \to \forall x \psi}\ \forall\text{I}$$

(where x is not free in φ in $\forall$I)

Note that $\Box$I: $x \approx y \to \Box(x \approx y)$ is provable based on I1, N, I2.

CBF and BF

Theorem

$\vdash_{QK} \Box\forall x\varphi \rightarrow \forall x\Box\varphi$ *(CBF)*

Proof.

1. $\forall x\varphi \rightarrow \varphi$ (∀E)
2. $\Box(\forall x\varphi \rightarrow \varphi)$ (N,1)
3. $\Box\forall x\varphi \rightarrow \Box\varphi$ (K,MP,2)
4. $\Box\forall x\varphi \rightarrow \forall x\Box\varphi$ (∀I,3)

□

CBF and BF

Recall normal modal logic **B** features axiom B: $\varphi \to \Box\Diamond\varphi$.

Theorem

$\vdash_{QB} \forall x \Box\varphi \to \Box\forall x\varphi$ *(BF)*

1. $\forall x\Box\varphi \to \Box\Diamond\forall x\Box\varphi$ (B)
2. $\forall x\Box\varphi \to \Box\varphi$ ($\forall$E)
3. $\Diamond\forall x\Box\varphi \to \Diamond\Box\varphi$ (Mono$\Diamond$)
4. $\Diamond\Box\varphi \to \varphi$ (B)
5. $\Diamond\forall x\Box\varphi \to \varphi$ (3,4)
6. $\Diamond\forall x\Box\varphi \to \forall x\varphi$ ($\forall$I,5)
7. $\Box\Diamond\forall x\Box\varphi \to \Box\forall x\varphi$ (Mono$\Box$,6)
8. $\forall x\Box\varphi \to \Box\forall x\varphi$ (1,7)

This means **QB=QB+BF=QB+BF+CBF**, where **B** can be replaced by **S5** for example. BTW **QB** also proves $\Box$NI.

Soundness and completeness in General

Theorem (Soundness)

If $\mathcal{F}$ is a frame for a normal propositional modal logic **S**, *then $\mathcal{F}$ also validates* **QS+BF**.

For completeness, we prove the equivalent statement that each consistent set of formulas is satisfiable in a proper model.

Theorem (Completeness)

Suppose that the frame of the canonical model for **QS+BF** *is a frame for* **S**. *Then* **QS+BF** *is complete w.r.t. the class of all frames for* **S**.

Corollary (Completeness)

QS+BF *is sound and complete w.r.t. the class of all frames for* $S \in \{K, T, K4, S4, S5\}$.

What is the canonical model for **QS+BF**?

Completeness of System QS+BF

Recall the strategy of the completeness proof for normal propositional modal logic **S**:

- Prove that every **S**-consistent set Σ can be extended to a maximal consistent set Σ^+. (Lindenbaum's Lemma)
- Use maximal consistent sets as worlds in W^c in $\mathcal{M}^c$ which ensures Σ belongs to some state in $\mathcal{M}^c$.
- Define R^c of $\mathcal{M}^c$ as wR^cu if $\Box\varphi \in w$ then $\varphi \in u$ and define V^c as $V^c(p) = \{w \in W^c \mid p \in w\}$.
- Prove that for every $\Diamond\varphi \in w$ there is a successor v of w such that $\varphi \in v$ (Existence Lemma)
- For any formula φ, $\mathcal{M}^c, w \models \varphi$ iff $\varphi \in w$. (Truth Lemma)
- Each **S**-consistent set is then satisfiable in the canonical model. Verify the canonical frame validates the logic **S**.

Completeness of System QS+BF

Recall the strategy of the completeness proof for FOL (call it **Q**):

- We also need a canonical model $\mathfrak{A}$ and an assignment σ with the truth lemma.
- For the quantifier case to go through, we need to use Henkin set where each $\exists x\varphi$ formula in it also has a witness $\varphi[y/x]$. This is can be done by requiring the so-called $\forall$-property: for each $\forall x\varphi$, $\varphi[y/x] \to \forall x\varphi$ is in the set for some y.
- Prove that every **Q**-consistent set Γ can be extended to a maximal consistent Henkin set Λ by extending the language with new symbols to provide witnesses.
- Let D be the set of equivalence classes over equality $\approx$ in Λ. Let $I(P^n) = \{\langle [x_1], \cdots, [x_n] \rangle \mid P(x_1, \cdots, x_n) \in \Lambda\}$.
- Every **Q**-consistent set is satisfiable in the canonical model.

Completeness of System QS+BF

The canonical model of **QS+BF** is based on combining the two strategies together:

$$\mathcal{M}^c = \langle W^C, R^C, D^C, I^C \rangle$$

where:

- W^C is the set of all maximal consistent sets with the $\forall$-property in the extended language $\mathcal{L}_{\approx}^{\Sigma^+}$ with new variables.
- $wR^C v$ iff $\Box\varphi \in w$ implies $\varphi \in v$ for any φ in $\mathcal{L}_{\approx}^{\Sigma^+}$.
- $D^C = \{[x] \mid x \text{ is a variable in } \mathcal{L}_{\approx}^{\Sigma^+}\}$.
- For each $w \in W^C$, $I^C(w)$ is a function such that $\langle [x_1], \cdots, [x_n] \rangle \in I^C(P^n)$ iff $P^n(x_1 \cdots x_n) \in w$.

Wait, I cheated a bit: we actually need to cut out the connected part given a world w such that $x \approx y \in w$ iff $x \approx y \in v$ for any other v in the model (by $\Box$NI). This makes $[x]$ well-defined.

Recall System QS+BF: merging FOL and a ML

Definition (Axioms of QS+BF)

Given a normal propositional modal logic S, QS is defined as:

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- I1 $x \approx x$
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- □NI $x \not\approx y \rightarrow \Box(x \not\approx y)$
- ∀E $\forall x \varphi \rightarrow \varphi[y/x]$
- BF $\forall x \Box \varphi \rightarrow \Box \forall x \varphi$

Rules:

$$\frac{\varphi \rightarrow \psi \quad \varphi}{\psi}\ \text{MP} \qquad \frac{\vdash \varphi}{\vdash \Box \varphi}\ \text{N} \qquad \frac{\vdash \varphi \rightarrow \psi}{\vdash \varphi \rightarrow \forall x \psi}\ \forall\text{I}$$

(where x is not free in φ in ∀I)

What role does BF play?

Lemma (Existence of a witnessed successor)

*Γ is a maximal **QS+BF**-consistent set of $\mathcal{L}^{\Sigma^+}_{\approx}$-formulas with $\forall$-property, φ is a formula such that $\Diamond\varphi \in \Gamma$, then there is a consistent set Δ of $\mathcal{L}^{\Sigma^+}_{\approx}$-formulas with $\forall$-property such that $\{\chi \mid \Box\chi \in \Gamma\} \cup \{\varphi\} \subseteq \Delta$. Δ can be extended into a successor.*

We enumerate all $\mathcal{L}^{\Sigma^+}_{\approx}$-formulas in form $\forall x\varphi$ as $\forall x_1\varphi_1, \forall x_2\varphi_2, \cdots$, and define a sequence of formulas $\psi_1, \psi_2, \cdots$ as:

$$\psi_0 = \varphi$$

$$\psi_1 = \psi_0 \wedge (\varphi_1[y_1/x_1] \to \forall x_1\varphi_1)$$

$$\cdots$$

$$\psi_{n+1} = \psi_n \wedge (\varphi_{n+1}[y_{n+1}/x_{n+1}] \to \forall x_{n+1}\varphi_{n+1})$$

where for every $n \geq 1$, y_{n+1} is the first variable such that

What role does BF play?

We can show $\{\chi \mid \Box\chi \in \Gamma\} \cup \{\varphi\}$ is consistent as in PML. We show there always will be a y_{n+1} satisfying $(*)$, which is based on BF (no fresh variable to use). Suppose not, there is a smallest $n \geq 0$ such that (1) $\{\chi \mid \Box\chi \in \Gamma\} \cup \{\psi_n\}$ is consistent but (2) for every y in $\mathcal{L}^{\Sigma^+}_{\approx}$ there is some $\{\Box\chi_1, \cdots, \Box\chi_k\} \subseteq \Gamma$ such that

$$\vdash (\chi_1 \wedge \cdots \wedge \chi_k) \to (\psi_n \to \neg(\varphi_{n+1}[y/x_{n+1}] \to \forall x_{n+1}\varphi_{n+1}))$$

thus by monotonicity of $\Box$ and its distribution over conjunction:

$$\vdash (\Box\chi_1 \wedge \cdots \wedge \Box\chi_k) \to \Box(\psi_n \to \neg(\varphi_{n+1}[y/x_{n+1}] \to \forall x_{n+1}\varphi_{n+1}))$$

Since χ is a maximal consistent set, for every y we have:

$$\theta(y) = \Box(\psi_n \to \neg(\varphi_{n+1}[y/x_{n+1}] \to \forall x_{n+1}\varphi_{n+1})) \in \Gamma$$

Let z be a variable that does not occur in φ_{n+1} or ψ_n. By

What role does BF play?

Then

$$\forall z\theta(z) = \forall z\Box(\psi_n \to \neg(\varphi_{n+1}[z/x_{n+1}] \to \forall x_{n+1}\varphi_{n+1})) \in \Gamma$$

By BF,

$$\Box\forall z(\psi_n \to \neg(\varphi_{n+1}[z/x_{n+1}] \to \forall x_{n+1}\varphi_{n+1})) \in \Gamma$$

Since z does not occur in ψ_n, by first-order reasoning

$$\Box(\psi_n \to \forall z\neg(\varphi_{n+1}[z/x_{n+1}] \to \forall x_{n+1}\varphi_{n+1})) \in \Gamma$$

But since z also does not occur in φ_{n+1} first-order reasoning:

$$\vdash \neg\forall z\neg((\varphi_{n+1}[z/x_{n+1}] \to \forall x_{n+1}\varphi_{n+1}))$$

We have $\Box\neg\psi_n \in \Gamma$ thus $\{\chi \mid \Box\chi \in \Gamma\} \cup \{\psi_n\}$ is inconsistent. Contradiction! Therefore, we can make a union of all ψ_n and make sure it is still consistent, which is the consistent set we need to build a witnessed successor given $\Diamond\varphi \in \Gamma$.

Incompleteness

Given a complete normal modal logic S (over the frames validating it), is QS+BF also complete?

The answer is no. For example, let S4M= S4+$\Box\Diamond\varphi \to \Diamond\Box\varphi$, it is complete over its frames (reflexive, transitive, and atomic).

QS4M+BF is not sound and complete over *any* class of frames. There are valid formulas involving quantifiers and modalities that are not provable in QS4M+BF: the same property can be defined by PML formulas and FOML-formulas, but they are not provably equivalent in the system.

The above incompleteness is fixable by adding extra axioms, but there are cases that cannot be fixed in this way.

BTW Craig interpolation does not hold for QS+BF for the usual S.

In case you don't like BF

$$\forall x \Box \varphi \to \Box \forall x \varphi$$

BF is valid over constant domain models, thus if you don't like it, you need to change the model (or the semantics).

This leads to popular models with increasing/expanding domains.

Kripke models with increasing domains

Definition (Increasing Domain Model)

A increasing domain model is a tuple $\mathcal{M} = \langle W, R, D, \delta, I \rangle$ where:

- W is a non-empty set (possible worlds);
- $R \subseteq W \times W$ (accessibility relation);
- D is non-empty global domain;
- δ is a function assigning each $w \in W$ a local domain $\delta(w) \subseteq D$ such that for all $w, v \in W$, $\delta(w) \subseteq \delta(v)$ if Rwv;
- For each $w \in W$, $I(w)$ is a function assigning a n-ary relation to P^n, i.e., $I(w)(P^n) \subseteq D^n$.

A constant-domain model satisfying $\delta(w) = D$ for all $w \in W$.

We also write D_w as abbreviations for $\delta(w)$.

Definition (Truth Conditions)

Given a model $\mathcal{M}$, world w in it, and an assignment σ:

$$\mathcal{M}, w, \sigma \models \forall x \varphi \quad \Leftrightarrow \quad \text{for all } a \in D_w, \mathcal{M}, w, \sigma(a/x) \models \varphi$$

φ is valid in a model $\mathcal{M}$ if for each worlds w each relevant assignment σ such that $\sigma(x) \in D_w$ for all $x \in Var$.

In this way, we will keep the FO validities like $\forall x P(x) \to P(y)$.

$\forall x \Box \varphi \to \Box \forall x \varphi$ is not longer valid over increasing domain models.

(CBF) $\Box \forall x \varphi \to \forall x \Box \varphi$ is still valid.

Axiomatizations and completeness proofs

Like in the case of constant-domain models, various **QS** are compelete w.r.t. the frame classes of **S**. Note that if **S** extends **B** then BF is provable in **QS** as we have shown. Also, the increasing domain condition and the symmetric frame together enforce a constant domain essentially.

Note that in most of the cases, we cannot use the BF in the proof, but the increasing domain condition leaves us more space in building the canonical model. The usual trick to facilitate a witnessed existence lemma is to associate to each world a local language that extends the original language $\mathcal{L}^{\Sigma}_{\approx}$ with infinitely many new variables as the witnesses but still leaves countably many variables unused in the extended $\mathcal{L}^{\Sigma^+}_{\approx}$. The local variables will help us to define the local domains and make sure they are increasing.

Other variants

What if you also don't like CBF? There are ways to handle it.

- Varying domain models models and semantics;
- Counterpart semantics: does not accept cross-world identity.

Disambiguation with extra machinery

The number of planets in solar system is necessarily even.

Two readings:

de dicto It is necessary that the number of planets in the solar system is even. Intuitively false!

de re Eight, the number of planets in the solar system, is necessarily even. Intuitively true!

If we have constants $c, d \cdots$ in the language, we can express the first by $\Box Even(c)$ where c is a non-rigid designator for the number of planets in the solar system.

How to express the second meaning using c and *Even*?

Disambiguation with extra machinery

There are various ways, for instance:

- λ-abstraction (Fitting): turn a formula with free variables into a predicate e.g., $\langle \lambda x.\Box Even(x)\rangle(c)$ for the *de re* reading;
- Assignment operator (Kooi): $[x := c]\Box Even(x)$;
- Indexed modality (Corsi and Orlandelli): $[^{c}_{x}]Even(x)$

Wait for Eugenio Orlandelli's talk!

Expressivity

Understanding the FOML expressivity over models

There is a translation from FOML language to the corresponding two-sorted FOL language $\mathcal{L}^{2SFOL}_{\approx}$:

$$\varphi ::= x \approx y \mid Q^{n+1}(ux_1 ... x_n) \mid Ruv \mid Eux \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \forall x\varphi \mid \forall u\varphi$$

where $x, y \in Var$ and $u, v \in$ War which is a collection of *world variables* disjoint from *Var*; each n-ary P^n there is a unique $n+1$-ary Q_P. E is a new predicate symbol to say which object exists on which world. We can define the translation:

$$\begin{aligned} t_u(x \approx y) &= x \approx y & t_u(P\bar{x}) &= Q_P(u, \bar{x}) \\ t_u(\neg\psi) &= \neg t_u(\psi) & t_u(\varphi \wedge \psi) &= t_u(\varphi) \wedge t_u(\psi). \\ t_u(\Box\psi) &= \forall v(Ruv \to t_v(\psi)) & t_u(\forall x\psi) &= \forall x(Eux \to t_u(\psi)) \end{aligned}$$

We can show $\mathcal{M}, w, \sigma \vDash \varphi$ iff $\mathcal{M}, \sigma(w/u) \Vdash_{2SFOL} t_u(\varphi)$

Given $\mathcal{M}$ and $\mathcal{N}$, non-empty relation $Z \subseteq (W^{\mathcal{M}} \times D^*_{\mathcal{M}}) \times (W^{\mathcal{N}} \times D^*_{\mathcal{N}})$ is call an *world-object-bisimulation*, if for every $((w\overline{a}), (v\overline{b})) \in Z$ such that $|\overline{a}| = |\overline{b}|$ the following holds:

PISO $\overline{a}$ and $\overline{b}$ form a partial isomorphism w.r.t. identity and interpretations of predicates at w and v respectively.

$\exists$Zig For any $c \in D^{\mathcal{M}}_w$, there is a $d \in D^{\mathcal{N}}_v$ such that $w\overline{a}cZv\overline{b}d$.

$\Diamond$Zig for any $w' \in W^{\mathcal{M}}$ if wRw' then there exists v' in $W^{\mathcal{N}}$ such that vRv' and $w'\overline{a}Zv'\overline{b}$.

vice versa for $\exists$Zag and $\Diamond$Zag.

We say $\mathcal{M}, w\overline{a}$ and $\mathcal{N}, v\overline{b}$ are wo-*bisimilar* ($\mathcal{M}, w\overline{a} \leftrightarroweq_{wo} \mathcal{N}, v\overline{b}$) if $|a| = |b|$ and there is an wo-bisimulation linking $w\overline{a}$ and $v\overline{b}$.

Van Benthem Characterization

Theorem (Invariance Theorem)

$\mathcal{L}_{\approx}$ is invariant under world-object bisimulations.

Theorem (Characterization Theorem)

The following statements are equivalent for formulas in $\mathcal{L}_{\approx}^{2SFOL}$:

(a) *φ is invariant for world-object bisimulations.*

(b) *φ is definable by a formula of $\mathcal{L}_{\approx}$.*

Expressivity over frames

PML over frames via validity is essentially a fragment of MSO, as there is implicit second-order quantification underneath validity.

FOML is even stronger over frames. Various results for PML generalize to FOML:

- Sahlqvist fragment
- Goldblatt-Thomason theorem

There are FOML-definable frame classes that are not PML-definable.

Wait for Reihane Zoghifard's talk!

Computational aspects

Finding decidable fragments of FOML is very hard

Simply putting a decidable fragment of first-order logic plus a modality does not work at all.

Language	Decidability	Ref
P^1	undecidable	[Kripke 62]
x, y, p, P^1	undecidable	[Gabbay 93]
$x, y, \Box_i$, single P^1	undecidable	[Rybakov & Shkatov 17]

The decidable fragments are rare (only one x in $\Box$).

Language	Decidability	Ref
single x	decidable	[Segerberg 73]
$x, y / P^1 /$ GF, $\Box_i(x)$	decidable	[Wolter & Zakharyaschev 01]

It looks hopeless!

We can do better!

We need to exploit the unique feature of FOML: the interaction of quantifiers and modalities!

We can somehow repeat the secret of success of PML by restricting the occurrence of the quantifies to occur in combination with modalities, and this is the core idea of the Bundled Fragments.

Definition ($\exists\Box$-fragment (Wang 17))

$$\varphi ::= P\overline{x} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x\Box\varphi$$

It is not an ad-hoc restriction just for decidability but inspired by applications such as epistemic logics of know-wh, where $\exists x\Box$ and other bundles are abundant.

Bundled fragments

- $\exists\Box$ fragment is decidable over both increasing and constant domain models! $\forall x\Diamond$ weakens the power of $\forall$!
- A satisfiable $\exists\Box$ formula has a *finite tree* model.
- We have a tableau method for satisfiability checking
- Satisfiability checking of $\exists\Box$ fragment is PSPACE-complete (exactly as the complexity of propositional model logic)

Note that we do not need to restrict the arity of the predicates or the number of variable occurrences at all.

Wait for Anantha Padmanabha's talk!

Related areas

- Propositionally quantified modal logic
- Term-modal Logic
- Intensional Modal Logic
- Modal syllogistics and extensions
- Epistemic logics of know-wh
- Arbitrary announcement logic
- ...

Conclusions

Through worlds of truth and possibility,
Quantifiers dance with necessity.
Rigid designators hold their ground,
While predicates shift their meaning round.
In boxes, diamonds, ways things could be—
Kripke's frames stretch infinitely.
Barcan's formula stirs our minds to sate:
As existence and essence interrelate.

– Claude Sonnet 3.5

There are lots of interesting things to be explored for FOML!