Bundles in Non-classical Logics

Knowing How to Understand Intuitionistic Logic

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Intuitionistic Logic as Epistemic Logic of Knowing How

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The world of non-classical logics

Like classical logic but with “strange” things, for good reasons.

Relevant logic, Multi-valued logic, Intuitionistic logic,
Paraconsistent logic, Non-monotonic logic, Quantum logic ...

Intuitionistic logic

Rooted in Brouwer's intuitionism of philosophy of mathematics, but has its own life. Heyting went through the usual theorems of classical logic and picked some ones according to Brouwer's idea...

$$\alpha \to (\beta \to \alpha)$$
$$(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$$
$$\alpha \to (\beta \to (\alpha \wedge \beta))$$
$$(\alpha \wedge \beta) \to \alpha$$
$$(\alpha \wedge \beta) \to \beta$$
$$\alpha \to (\alpha \vee \beta)$$
$$\beta \to (\alpha \vee \beta)$$
$$(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$$
$$(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$$
$$\neg\alpha \to (\alpha \to \beta)$$

MP

Not provable in intuitionistic logic

- $\alpha \lor \neg\alpha$
- $\neg\neg\alpha \to \alpha$
- $\neg(\alpha \land \beta) \to \neg\alpha \lor \neg\beta$
- $(\alpha \to \beta) \to \neg\alpha \lor \beta$
- $(\alpha \to \beta \lor \chi) \to (\alpha \to \beta) \lor (\alpha \to \chi)$
- $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$

Early results and semantics

- Disjunction property
- Double negation translations of classical logic into it
- Embedding into S4 modal logic

Various semantics as technical tools:

- Algebraic semantics
- Topological semantics
- Realizability semantics
- Kripke semantics
- ...

Current picture

It has many surprising connections to other fields beyond logic and inspired various theories.

- In logic: Intermediate logic, Intuitionistic X logics ...
- In Math: forcing, constructive math, Heyting algebra
- In TCS: Curry-Howard correspondence, intuitionistic type theory (behind Coq and Lean), Verification tools ...
- In Philosophy: philosophy of language, epistemology, metaphysics, philosophy of logic/math...
- In AI: Intuitionistic fuzzy sets ...

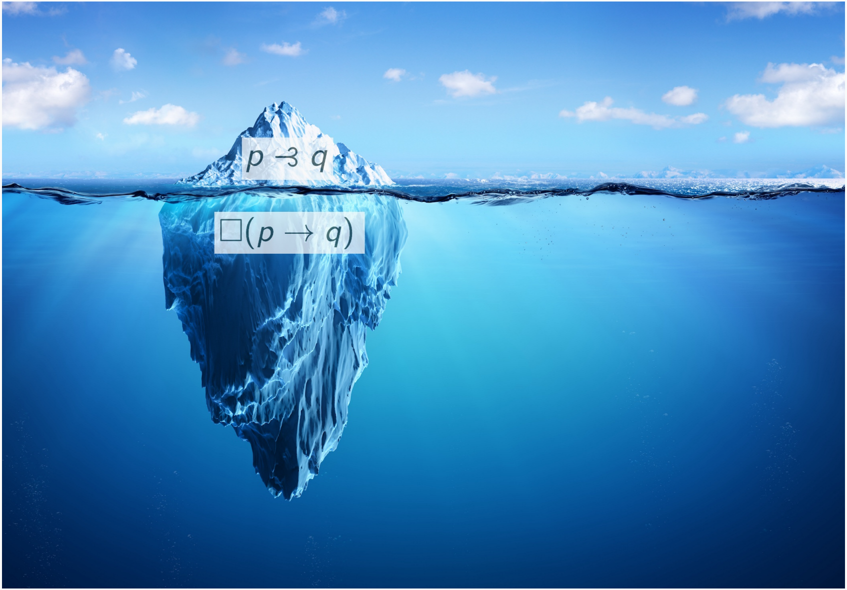
Do we really understand what intuitionistic logic is about? Why it is strange yet very useful?

Non-classical logics are typical icebergs for me...

Above water: syntactic behaviour; Below: semantic structures

Semantics for me is not simply a tool for obtaining completeness...

“Non-classicality” is sometimes due to deeper structures



Intuitionistic logic is an intricate one

We will dive into the deep ocean of logic

And meet some of the greatest minds

Make the **hidden** information **explicit**

What is intuitionistic “truth”?

A crucial observation

Excursion into the history

Epistemic decoding/unbundling

Example: Inquisitive Logic & More

Conclusions

What is intuitionistic “truth”?

The counterpart of "truth" in intuitionistic logic

Classical logic is about truth-preserving reasoning.

Intuitionistic logic is about ????????-preserving reasoning?

What does an intuitionistic formula say *intuitively*? E.g., what do "invalid" $\alpha \vee \neg\alpha$ or $\neg\neg\alpha \to \alpha$ express in intuitionistic logic?

The same formula may mean quite different things compared to the classical setting.

The conception of "intuitionistic truth" may also be the key to its surprising usefulness.

Brouwer-Heyting-Kolmogorov (BHK) interpretation

BHK *proof* interpretation/explanation of connectives:

(H1) A proof of $\alpha \wedge \beta$ is given by presenting a proof of α and a proof of β

(H2) A proof of $\alpha \vee \beta$ is given by presenting either a proof of α or a proof of β

(H3) A proof of $\alpha \to \beta$ is a construction that transforms any proof of α into some proof of β

(H4) Absurdity $\perp$ has *no* proof.

$\neg\alpha$ is the abbreviation of $\alpha \to \perp$.

A proof of an atomic proposition *p* is given by presenting a mathematical construction in Brouwer's sense.

How to go from proof to "truth"?

What about intuitionistic "truth"?

Most people take intuitionistic truth $\approx$ provability, but what does *provability* mean?

- Actualist
  - There exists a proof (to be discovered).
  - Has been proved
- Possibilist:
  - Will be proved
  - Can be proved in principle
- Extra epistemic layer on top of them

But these philosophical discussions have little impact on the mathematical theories of intuitionistic logic. The naive actualist provability notion is consistent with classical logic.

Our goal

To understand intuitionistic logic
philosophically and mathematically,

we will try to make informal interpretation coincide
with the formal semantics.

What we will be talking about

- An observation
- A bit of history
- A bit of philosophy
- A bit of modal logic

Disclaimers

- For simplicity, we will focus on the propositional part.
- We will focus on the ideas instead of proofs.

A crucial observation

Kolmogorov's problem interpretation

Kolmogorov: intuitionistic logic is about solving problems.

Intuitively, each formula denotes a (type of) problem (instead of a proposition) which has a (possibly empty) set of solutions. The logical connectives are constructors to build complex problems based on simpler problems with computed sets of solutions, e.g. the solutions of $\alpha \to \beta$ are constructions turning each solution of α into some solution of β.

propositions	problems
proofs	solutions

Atomic propositions denote atomic problems.

Brouwer	Kolmogorov
Heyting	Yu. T. Medvedev

Logic of (finite) problems by Medvedev (1962)

Medvedev formalized the (BH)K-interpretation in terms of problems and solutions. A problem is a pair $\langle X, Y\rangle$ such that:

- X is a non-empty set (solution space / *"admissible possibilities"*);
- $Y \subseteq X$ is the set of actual solutions.

E.g., solving $x^2 = 1$ in $\mathbb{Z}$ is $\langle \mathbb{Z}, \{1, -1\}\rangle$. A *finite problem* is a problem where X is finite. Compound problems are defined:

$$\langle X_1, Y_1\rangle \wedge \langle X_2, Y_2\rangle = \langle X_1 \times X_2, Y_1 \times Y_2\rangle$$

$$\langle X_1, Y_1\rangle \vee \langle X_2, Y_2\rangle = \langle X_1 \sqcup X_2, Y_1 \sqcup Y_2\rangle$$

$$\langle X_1, Y_1\rangle \to \langle X_2, Y_2\rangle = \langle {X_2}^{X_1}, \{f\colon X_1 \to X_2 \mid f[Y_1] \subseteq Y_2\}\rangle.$$

where $X \sqcup X' = (X \times \{0\}) \cup (X' \times \{1\})$ (disjoint union).

Take $\langle X, Y\rangle$ as $\langle$proof space, actual proofs$\rangle$, we can see proof-interpretation is a special (non-finitary) case.

Logic of (finite) problems by Medvedev (1962)

Medvedev not only formalized the problem-interpretation but also gave a formal definition of truth in terms of *(uniform) solvability*.

Let j be an assignment giving a problem to each atomic p, e.g., $j(p) = \langle X, Y \rangle$, and we denote $j_1(p) = X$ and $j_2(p) = Y$. $j(\bot) = \langle \{\varnothing\}, \varnothing \rangle$. It can be extended to assign problems to any α. We write $j' \sim j$ if $j_1(p) = j'_1(p)$ for all atomic p: j and j' may only disagree on the actual solutions of the problems (solution space is certain).

A formula α is *"true"* (uniformly solvable) under j iff $\bigcap_{j' \sim j} j'_2(\alpha) \neq \varnothing$. i.e., α has a *uniform* solution given any $j' \sim j$.

α is Medvedev valid iff it is true under any *finite* assignment;
α is Skvortsov valid iff it is true under any assignments.

Example

$$j(p) = \langle X, Y \rangle, j(\bot) = \langle \{\varnothing\}, \varnothing \rangle$$

$$j(\neg p) = j(p \to \bot) = j(p) \to j(\bot) = \langle \{\varnothing\}^X, \{f : X \to \{\varnothing\} \mid f[Y] \subseteq \varnothing\} \rangle$$

$$j_2(\neg p) \neq \varnothing \text{ iff } j_2(p) = \varnothing \qquad j_2(\neg p) = \varnothing \text{ iff } j_2(p) \neq \varnothing$$

$$j(p \vee \neg p) = \langle j_1(p) \times \{0\} \cup j_1(\neg p) \times \{1\}, j_2(p) \times \{0\} \cup j_2(\neg p) \times \{1\} \rangle$$

$$\begin{aligned} j_2(p \vee \neg p) &= \begin{cases} j_2(\neg p) \times \{1\} & j_2(p) = \varnothing \\ j_2(p) \times \{0\} & j_2(p) \neq \varnothing \end{cases} \\ &= \begin{cases} \{\varnothing\}^X \times \{1\} & j_2(p) = \varnothing \\ Y \times \{0\} & j_2(p) \neq \varnothing \end{cases} \end{aligned}$$

No uniform solution for $p \vee \neg p$ in general.

Medvedev's logic

- The set of Medvedev valid formulas forms an intermediate logic, containing some more valid formulas than IPC
- Kripke semantics based on non-empty subsets of a finite set
- Lots of open problems:
  - Axiomatization
  - Decidability
  - The same as Skvortsov logic?
  - ...

People seem to forgot Medvedev's original semantics...

The uniformity is important!

$\bigcap_{j' \sim j} j'_2(\alpha) \neq \varnothing$ is not the same as $\forall j' \sim j : j'_2(\alpha) \neq \varnothing$.

Similar ideas appeared **not only** in Medvedev's work:

- Math: e.g., in Läuchli (1970) On a complete semantics for standard intuitionistic predicate logic; Friedman (2000) for propositional intuitionistic logic...
- CS: e.g., Constable and Bickford (2014) Intuitionistic completeness of first-order logic.
- Details matter: treatment of $\perp$, the class of eligible functions and so on.

What exactly is this *uniformity*?

The crucial observation

α is true under j iff $\bigcap_{j' \sim j} j'_2(\alpha) \neq \varnothing$.

α is true under j iff $\exists x \forall j' \sim j, x \in j'_2(\alpha)$.

Wait! It looks pretty much like the *bundled modality* $\exists x \mathcal{K}$ in my approach for know-wh logics...

α is true under j iff $\exists x \mathcal{K}$(x *is a proof/solution of* α).
(given *full* uncertainty w.r.t. j).

α is true under j iff *knowing a proof/solution of* α.

α is true under j iff *knowing how to prove/solve* α

The conception of intuitionistic truth

Intuitionistic truth of α = knowing how to prove/sovle α

Is it entirely a new idea?

Let's take an excursion into the history.

Excursion into the history

Related ideas in the past 100 years

Surprisingly, it was almost exactly **what Heyting said 95 years ago** in his first published explanation of intuitionistic logic [Heyting 30, translation in Mancosu 98].

> *To satisfy the intuitionistic demands, the assertion must be the realisation of the expectation expressed by the proposition p. Here, then, is the Brouwerian assertion of p:* ***It is known how to prove*** p**.** *We will denote this by* $\vdash p$*. The words "to prove" must be taken in the sense of "to prove by construction".*

He elaborated the epistemic interpretation in depth 26 years later on in...

A largely forgotten paper (in French) by Heyting

A. Heyting. La conception intuitionniste de la logique. *Les études philosophiques*, vol. 11 (1956), pp. 226–233.

The English translation *The Intuitionistic Conception of Logic* was drafted by Claude-3.5-Sonnet and carefully edited by Philippe Balbiani, Hans van Ditmarsch, Dick de Jongh and Yanjing Wang.

Thanks to the publisher's kind approval, you can download it for free from the link below or simply google it
`https://logic.pku.edu.cn/xzdt/xjxx/540525.htm`

A largely forgotten paper (in French) by Heyting

Heyting starts the article with:

> *Logic is often studied as a purely formal science, where the concern is not with the meaning of logical notions, but only with their formal properties. Therefore, one does not ask what it means for a proposition to be true or false, but only deals with the formal conditions under which one proposition can be deduced from other propositions. However, as soon as one wants to apply logic, one must address the question of the meaning of the word "true" and other logical terms [...]*

The main point of the paper is to argue the intuitionistic conception of logic is a logic of *knowing* instead of a logic of *being* (classical logic).

Distinctions between being and knowing: double negation

In the logic of *being*, the following are equivalent.

- A There is a counterexample to Goldbach's conjecture.
- B It is not the case that there is no such a counterexample.

In the logic of *knowing*, the following are clearly different.

- C I know a counterexample.
- D I have reached a contradiction from the assumption that there is no counterexample.

Distinctions about excluded middle

In the logic of being, one of the following is true:

E There is a counterexample to Goldbach's conjecture.

F There is no counterexample.

In the logic of knowledge, both can be false:

G I know how to calculate a counterexample.

H I know how to deduce a contradiction from the hypothetical assumption that we have found a counterexample.

There is no reason to say that either (G) or (H) must be true.

Other related ideas in the past 100 years

Martin-Löf (1985) was also explicit about it in developing the intuitionistic type theory:

> *Observe that knowledge of a judgement of the second form [**A is true**] is **knowledge-how**, more precisely, **knowledge how to verify A**, whereas knowledge of a judgement of the first form [A is a proposition] is knowledge of a problem, expectation, or intention, which is knowledge what to do, simply.*

Martin-Löf is another former student of Kolmogorov.

Related ideas in the past 100 years

Intuitionistic logic as Epistemic Logic (Hintikka 2001):

> *For instance, Brouwer's "counter-examples" to the law of excluded middle are blatantly in terms of* ***what is known, not of what is the case****.*

> *The most fundamental feature of the diagnosis is that the key notion of the intuitionists turns out to be,* ***not our knowledge of mathematical truths, but our knowledge of mathematical objects****, prominently including our knowledge of the identity of functions. The crucial notion, in other words,* ***is not knowing that but knowing what (which, who, where, $\cdots$), in brief knowing + an indirect question, that is, knowledge of objects rather than knowledge of truths****.*

Anticipation of an epistemic approach

Hintikka (2001) says:

> *All told, there is unmistakably an epistemic element in the intuitionistic way of thinking. And what makes that observation timely is that* ***an opportunity of implementing that epistemic element by means of an explicit epistemic logic has just been opened.***

Anticipation of an epistemic approach

In *Implicit and Explicit Stances in Logic*, Van Benthem (2019):

> *The epistemic logic for semantic information is S5, while the Gödel translation into S4 reflects a view of intuitionistic models as temporal processes of inquiry. Thus,* ***an explicit counterpart to intuitionistic logic needs a temporal version of dynamic epistemic logic*** *[...] A technical implementation would be an embedding of S4 into a bimodal temporalized S5,*

Recent ideas

Ciardelli and Roelofsen (2011) on inquisitive logic:

> *Traditionally, an information state s is taken to support a formula φ iff it is known in s that φ is true. This is not how support should be thought of in the present setting. However, there is a closely related interpretation that is appropriate: $s \vDash \varphi$ can be read as stating the conditions under which* ***it is known in s how*** *φ is realized.*

However, Ciardelli abandoned this idea in later publications.

More recent ideas from Melikhov

Melikhov (2013-18) made a "mathematician's attempt to understand intuitionistic logic" using Paulson's higher-order meta-logic, and pinpointed the distinction between knowledge-that and knowledge-how inspired by Kolmogorov on **Hilbert's and Brouwer's** mathematics :

> *The two sides of mathematics referred to by Kolmogorov can be seen as representing two modes of knowledge (including formalized mathematical knowledge, but also keeping in mind subjects such as common knowledge and collective intelligence):*
>
> - ***knowledge-that*** *(or knowledge of truths)*
> - ***knowledge-how*** *(or knowledge of methods).*

And...

Although the distinction between knowledge-that and knowledge-how is made explicitly and often used as "informal semantics" for the joint logic, Melikhov (2018) remarked:

> *There is, however,* ***hardly any connection with the distinction made in philosophy between "knowledge how" and "knowledge that"*** *in the tradition originating with G. Ryle, whose "knowledge how" is an unconscious, non-articulable ability.*

A bit of philosophy can actually help...

A bit of philosophy and philosophical logic do help

Certain types of know-how can be formalized and understood!

- Inspired by linguistic evidence, philosophers try to understand knowing how using (quantified) knowing that, e.g., Stanley and Williamson (2001).
- Logical structure coincides with what Hintikka pioneered on the *de re* knowledge-wh using first-order modal logic.
- These led to my "bundled" treatment of knowing how and other know-wh, which is the missing tool to capture Heyting's original idea about intuitionistic truth.

Epistemic decoding/unbundling

Intuitionistic and intermediate logics: tough nuts to crack

You need a tool

You need a *right* tool

Decoding intuitionistic and intermediate logics

The tool: **dynamic epistemic logic of *knowing how*** as our looking glasses.

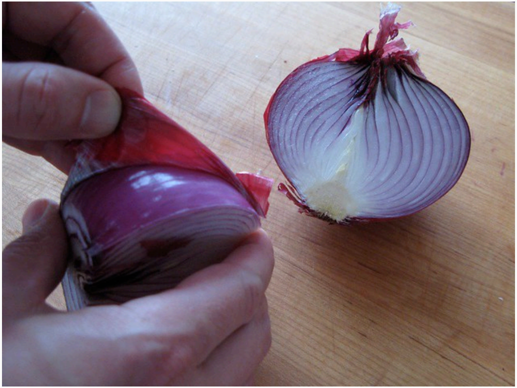
The general process:

- First turn each intuitionistic formula α into $\mathcal{K}h\alpha$
- Try to "open up" the $\mathcal{K}h$ formulas (to make the meaning of $\mathcal{K}h\alpha$ explicit) by using:
  - classical connectives $\vee, \wedge, \rightarrow$
  - classical atoms *p*
  - normal know-that operator $\mathcal{K}$
  - dynamic operator
  - Eventually $\vDash \mathcal{K}h\alpha \leftrightarrow \varphi$ where φ is almost-free of $\mathcal{K}h$.
- Try to axiomatize the full logic

Ideally, I would like to walk you through the process of adding those constructs and modalities gradually...

It is like peeling the onion step by step.

Taking meta language notions into the object language and you can see more.



However, we don't have the time for it, so let's go straight to the full language and focus on the important ideas.

A dynamic epistemic language

Definition (Language of DELKh)

Given **P**, the language of Dynamic Epistemic Logic of Knowing How (**DELKh**) is defined as follows:

$$\alpha ::= \bot \mid p \mid (\alpha \vee \alpha) \mid (\alpha \wedge \alpha) \mid (\alpha \to \alpha)$$

$$\varphi ::= \bot \mid p \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi) \mid \Box\varphi \mid \mathcal{K}\varphi \mid \mathcal{K}h\alpha$$

where $p \in \mathbf{P}$.

The models

Recall that each assignment j assigns p a problem $\langle X, Y \rangle$ where $j_1(p) = X$ and $j_2(p) = Y$.

A **model** $\mathcal{J}$ for **P** is simply **a collection of indistinguishable assignments** for **P**, i.e., for any $j, j' \in \mathcal{J}$, $j_1(p) = j'_1(p)$ for each p.

Essentially **an epistemic (S5) model** with problem assignments of the basic propositions instead of valuations. This captures your uncertainty about the actual solutions/proofs.

Given $j \in \mathcal{J}$, $(\mathcal{J}, j)$ is a pointed model. A **submodel** of $(\mathcal{J}, j)$ is the pointed model $(\mathcal{J}', j)$ such that $j \in \mathcal{J}' \subseteq \mathcal{J}$.

Semantics

Definition (Semantics, connectives outside $\mathcal{K}h$ are classical)

$\mathcal{J}, j \vDash \bot$	$\Leftrightarrow$	never
$\mathcal{J}, j \vDash p$	$\Leftrightarrow$	$j_2(p) \neq \varnothing$ (p is solvable/provable)
$\mathcal{J}, j \vDash (\varphi \wedge \psi)$	$\Leftrightarrow$	$\mathcal{J}, j \vDash \varphi$ and $\mathcal{J}, j \vDash \psi$
$\mathcal{J}, j \vDash (\varphi \vee \psi)$	$\Leftrightarrow$	$\mathcal{J}, j \vDash \varphi$ or $\mathcal{J}, j \vDash \psi$
$\mathcal{J}, j \vDash (\varphi \to \psi)$	$\Leftrightarrow$	$\mathcal{J}, j \nvDash \varphi$ or $\mathcal{J}, j \vDash \psi$
$\mathcal{J}, j \vDash \mathcal{K}\varphi$	$\Leftrightarrow$	for all $j' \sim j, \mathcal{J}, j' \vDash \varphi$
$\mathcal{J}, j \vDash \mathcal{K}h\alpha$	$\Leftrightarrow$	$\exists x$ for all $j' \sim j, x \in j'_2(\alpha)$
$\mathcal{J}, j \vDash \Box\varphi$	$\Leftrightarrow$	for any submodel $\mathcal{J}'$ of $(\mathcal{J}, j), \mathcal{J}', j \vDash \varphi$

$j(\alpha)$ is defined by extending $j(p)$ under BHK interpretations.

$\Box$ is the dynamic operator modelling the information updates, to be used to decode the intuitionistic implication.

The point of other connectives and modalities

- $\mathcal{K}h\alpha$ captures intuitionistic truth of α.
- The apparatus in the language are used to decode $\mathcal{K}h\alpha$.
- In the end we want to reveal intuitively and explicitly the hidden meaning of intuitionistic α.

Warning: $\{\alpha \mid \vDash \mathcal{K}h\alpha\}$ is the Skvortsov logic. If restricted to finite problem then it is Medvedev's logic. To get back to standard intuitionistic logic we need to allow proofs for $\bot$ (Lauchli, Friedman), which departs form the BHK-interpretation:

$$j(\bot) = \langle \{\varnothing\}, \varnothing \rangle$$

Which should be intuitionistic logic?

We stick to the authentic BHK-setting where $\bot$ does not have proofs (and thus $j_2(\neg\alpha) \neq \varnothing$ iff $j_2(\alpha) = \varnothing$.)

Difference between $\mathcal{K}\alpha$ and $\mathcal{K}h\alpha$

Proposition

For any $\alpha \in \mathbf{PL}$ we have: $\mathcal{J}, j \vDash \alpha \Leftrightarrow j_2(\alpha) \neq \varnothing$

α is (classically) true means it is provable/solvable. We can then show that $\mathcal{K}\alpha$ has an equivalent semantics:

$\mathcal{J}, j \vDash \mathcal{K}\alpha$	$\Leftrightarrow$	for all $j' \sim j, \mathcal{J}, j' \vDash \alpha$
$\mathcal{J}, j \vDash \mathcal{K}\alpha$	$\Leftrightarrow$	for all $j' \sim j, \exists x, x \in j'_2(\alpha)$
$\mathcal{J}, j \vDash \mathcal{K}h\alpha$	$\Leftrightarrow$	$\exists x$ for all $j' \sim j, x \in j'_2(\alpha)$

Exactly the distinction between *de re* and *de dicto*:
knowing that α is provable vs. *knowing how to prove α.*

Objective provability + BHK is just classical logic.

Based on the above observation, we can build connection between $\mathcal{K}$ and $\mathcal{K}h$.

The interactions between $\mathcal{K}$ and $\mathcal{K}h$

Proposition

The following are valid:

$$\mathcal{K}h\alpha \to \mathcal{K}\alpha$$
$$\mathcal{K}h\alpha \to \mathcal{K}\mathcal{K}h\alpha$$
$$\neg\mathcal{K}h\alpha \to \mathcal{K}\neg\mathcal{K}h\alpha$$
$$\mathcal{K}\neg\alpha \to \mathcal{K}h\neg\alpha$$

The first one is an analogy of the intuitionistic epistemic logic axiom $\alpha \to \mathcal{K}\alpha$. The last one is a crucial one due to the standard BHK interpretation of $\bot$ as absurdity with no proof.

Proposition

$$\vDash \mathcal{K}h\neg\alpha \leftrightarrow \mathcal{K}\neg\alpha, \vDash \mathcal{K}h\neg\neg\alpha \leftrightarrow \mathcal{K}\alpha.$$

Decoding the $\mathcal{K}h$

With the help of the classical connectives outside $\mathcal{K}h$, we can decode the intuitionistic truth recursively with the following validities:

$$\neg \mathcal{K}h \bot$$

$$\mathcal{K}h(\alpha \wedge \beta) \leftrightarrow (\mathcal{K}h\alpha \wedge \mathcal{K}h\beta)$$

$$\mathcal{K}h(\alpha \vee \beta) \leftrightarrow (\mathcal{K}h\alpha \vee \mathcal{K}h\beta)$$

$$\mathcal{K}h(\alpha \to \beta) \to (\mathcal{K}h\alpha \to \mathcal{K}h\beta)$$

$$\mathcal{K}h(\alpha \to \beta) \leftrightarrow \mathcal{K}\Box(\mathcal{K}h\alpha \to \mathcal{K}h\beta)$$

The last one (if you allow all functions) unifies three notions of knowledge: as **range, dependency and procedure**.

Proposition (know-how preserving reasoning)

$\mathcal{K}h\alpha \vDash \mathcal{K}h\beta$ *iff* $\vDash \mathcal{K}h\alpha \to \mathcal{K}h\beta$ *iff* $\vDash \mathcal{K}h(\alpha \to \beta)$.

Properties of $\Box$

$\Box$ is at least normal S4, but with further properties:

$$\alpha \to \Box\alpha$$

$$\mathcal{K}h\alpha \to \Box\mathcal{K}h\alpha$$

$$\Diamond\Box\varphi \to \Box\Diamond\varphi$$

$$\Box\Diamond\varphi \to \Diamond\Box\varphi$$

$$\mathcal{K}\Box\varphi \to \Box\mathcal{K}\varphi$$

$$\Diamond \bigwedge_{0\leq i\leq n-1} \mathcal{K}h(\alpha_i \vee \neg\alpha_i)$$

$$\alpha \to \Diamond\mathcal{K}h\alpha$$

The last axiom (in contrapositive) reflects Brouwer's PIN principle (form perpetual ignorance to negation) for the "creating subject" and, Dummet: truth is potentially verifiable, Hilbert: We must know, we will know.

So what?

Does it help us to understand better?

Does it bring new insights?

Making things more transparent

- Meaning of formulas
- Intuitive validity/invalidity
- Double negation translation
- Disjunction property
- Kripke semantics

Intuitive reading of formulas

Excluded middle:

$\alpha \vee \neg\alpha$ in IntL

$\mathcal{K}h(\alpha \vee \neg\alpha)$

$\mathcal{K}h\alpha \vee \mathcal{K}h\neg\alpha$

$\mathcal{K}h\alpha \vee \mathcal{K}\neg\alpha$ (invalid)

Weak excluded middle:

$\neg\alpha \vee \neg\neg\alpha$ in IntL

$\mathcal{K}h(\neg\alpha \vee \neg\neg\alpha)$

$\mathcal{K}h\neg\alpha \vee \mathcal{K}h\neg\neg\alpha$

$\mathcal{K}\neg\alpha \vee \mathcal{K}\alpha$ (invalid)

Recap: Decoding the $\mathcal{K}h$

With the help of the classical connectives outside $\mathcal{K}h$, we can decode the intuitionistic truth recursively with the following validities:

$$
\begin{aligned}
&\neg \mathcal{K}h \bot \\
\mathcal{K}h(\alpha \wedge \beta) &\leftrightarrow (\mathcal{K}h\alpha \wedge \mathcal{K}h\beta) \\
\mathcal{K}h(\alpha \vee \beta) &\leftrightarrow (\mathcal{K}h\alpha \vee \mathcal{K}h\beta) \\
\mathcal{K}h(\alpha \to \beta) &\to (\mathcal{K}h\alpha \to \mathcal{K}h\beta) \\
\mathcal{K}h(\alpha \to \beta) &\leftrightarrow \mathcal{K}\Box(\mathcal{K}h\alpha \to \mathcal{K}h\beta)
\end{aligned}
$$

The last one (if you allow all functions) unifies three notions of knowledge: as **range, dependency and procedure**.

Intuitive reading of formulas

Based on our decoding:

our formula	meaning
$\mathcal{K}h(\alpha \vee \neg\alpha)$	$\mathcal{K}h\alpha \vee \mathcal{K}\neg\alpha$
$\mathcal{K}h(\neg\alpha \vee \neg\neg\alpha)$	$\mathcal{K}\neg\alpha \vee \mathcal{K}\alpha$

Intuitionistic law of excluded middle $\alpha \vee \neg\alpha$ is actually saying **either you know how to prove α or knowing it is unprovable**, of course should be invalid.The weak LEM $\neg\alpha \vee \neg\neg\alpha$ (you know whether α is provable) should not be valid either!

Another example of de Morgan law: $\neg(\alpha \wedge \beta) \to \neg\alpha \vee \neg\beta$

$$\vDash \mathcal{K}h(\neg(\alpha \wedge \beta) \to \neg\alpha \vee \neg\beta) \;\Leftrightarrow \vDash \mathcal{K}\neg(\alpha \wedge \beta) \to (\mathcal{K}\neg\alpha \vee \mathcal{K}\neg\beta)$$

Not valid!

Propositional Intuitionistic logic

You can try yourself to read the axioms now.

$$
\begin{aligned}
&\alpha \to (\beta \to \alpha) \\
&(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma)) \\
&\alpha \to (\beta \to (\alpha \land \beta)) \\
&(\alpha \land \beta) \to \alpha \\
&(\alpha \land \beta) \to \beta \\
&\alpha \to (\alpha \lor \beta) \\
&\beta \to (\alpha \lor \beta) \\
&(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \lor \beta) \to \gamma)) \\
&(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha) \\
&\neg\alpha \to (\alpha \to \beta)
\end{aligned}
$$

with MP rule.

Formalizing concepts

In the literature:

- α is decidable if and only if $\alpha \vee \neg\alpha$ is valid;
- α is stable (regular) if and only if $\neg\neg\alpha \to \alpha$ is valid.
- α is testable if and only if $\neg\alpha \vee \neg\neg\alpha$ is valid.

In our perspective:

- α is decidable if and only if $\mathcal{K}h\alpha \vee \mathcal{K}\neg\alpha$ is valid;
- α is stable (regular) if and only $\mathcal{K}\alpha \to \mathcal{K}h\alpha$ is valid.
- α is testable if and only if $\mathcal{K}\neg\alpha \vee \mathcal{K}\alpha$ is valid.

Kolmogorov mentioned that Brouwer already observed any $\neg\alpha$ is regular (thus behave classically), and consider intuitionistic logic an extension of classical logic.

Now we can look at the negative translations

There are various "double negation" translations from classical logic to intuitionistic logic by Glivenko, Gödel, Gentzen, Kolmogorov, such that:

$$\vdash_{\mathsf{CPC}} \alpha \text{ iff } \vdash_{\mathsf{IPC}} t(\alpha).$$

The simplest one (for propositional logic) is:

Theorem (Glivenko 1929)

$\vdash_{\mathsf{CPC}} \alpha$ *iff* $\vdash_{\mathsf{IPC}} \neg\neg\alpha$.

Now the semantic counterpart in our setting is trivial:

$$\vDash \alpha \text{ iff } \vDash \mathcal{K}\alpha \text{ iff } \vDash \mathcal{K}h\neg\neg\alpha.$$

We can understand other negative translations likewise.

Disjunction property

Proposition (Disjunction property)

If $\models \mathcal{K}h(\alpha \vee \beta)$ *then* $\models \mathcal{K}h\alpha$ *or* $\models \mathcal{K}h\beta$*.*

Note that $Kh(\alpha \vee \beta)$ is equivalent to $\mathcal{K}h\alpha \vee \mathcal{K}h\beta$. We can just show that $\not\models \mathcal{K}h\alpha$ and $\not\models \mathcal{K}h\beta$ implies $\not\models \mathcal{K}h\alpha \vee \mathcal{K}h\beta$.

Merging the two counter models for $\mathcal{K}h\alpha$ and $\mathcal{K}h\beta$ simply suffices.

Kripke semantics for Intuitionistic Logic

Inspired by McKinsey and Tarski's modal S4-translation.

A Kripke model for IPC $\mathcal{M}$ is $\langle S, \leq, V \rangle$ where

- S is a non-empty set of possible **states**;
- $\leq$ is a partial order over S;
- $V : S \to 2^{\mathbf{P}}$ assigns to each state some atomic propositions such that $p \in V(s)$ and $s \leq t \implies p \in V(t)$.

The truth conditions are given by the forcing condition:

$\mathcal{M}, s \Vdash p$	$\Leftrightarrow$	$p \in V(s)$
$\mathcal{M}, s \Vdash \alpha \wedge \beta$	$\Leftrightarrow$	$\mathcal{M}, s \Vdash \alpha$ and $\mathcal{M}, s \Vdash \beta$
$\mathcal{M}, s \Vdash \alpha \vee \beta$	$\Leftrightarrow$	$\mathcal{M}, s \Vdash \alpha$ or $\mathcal{M}, s \Vdash \beta$
$\mathcal{M}, s \Vdash \alpha \to \beta$	$\Leftrightarrow$	for all t such that $s \leq t : \mathcal{M}, t \Vdash \alpha \implies \mathcal{M}, t \Vdash \beta$
$\mathcal{M}, s \Vdash \neg\alpha$	$\Leftrightarrow$	for all t such that $s \leq t : \mathcal{M}, t \nVdash \alpha$

Kripke models are abstractions of the epistemic dynamics

From our perspective, Kripke models are *abstractions* of the update spaces of our epistemic models:

Kripke semantics	Our setting
a state	an (unpointed) S5 model
$\leq$	update relation
$s \Vdash p$	$\mathcal{K}hp$ is true
persistence of α	$\vDash \mathcal{K}h\alpha \to \Box\mathcal{K}h\alpha$
$s \Vdash \alpha \to \beta$	$\mathcal{K}\Box(\mathcal{K}h\alpha \to \mathcal{K}h\beta)$
$s \Vdash \neg\alpha$	$\mathcal{K}\Box\neg\mathcal{K}h\alpha$

It also explains why in the Kripke semantics of Medvedev's logic, we consider a powerset structure without the empty set.

How to get back to the standard intuitionistic logic?

You need to allow $\bot$ to have solutions and its solutions can also solve any α (the technical "compromise" by Lauchli, Friedman and others for completeness results of IL).

And in that case $\mathcal{K}h\neg\alpha \to \mathcal{K}\neg\alpha$ is still valid but not the other way around, neither is $\neg\mathcal{K}h\bot$!

We can focus on the simpler $\mathcal{K}h$-only language:

$$\alpha ::= \bot \mid (\alpha \vee \alpha) \mid (\alpha \wedge \alpha) \mid (\alpha \to \alpha)$$

$$\varphi ::= \bot \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi) \mid \mathcal{K}h\alpha$$

where $\neg\varphi := \varphi \to \bot$, $\neg\alpha := \alpha \to \bot$, $\top := \neg\bot$.

Standard Intuitionistic Logic as a Logic of Knowing How

Axioms beyond classical tautologies

1. $\mathcal{K}h(\alpha \to \beta) \to (\mathcal{K}h\alpha \to \mathcal{K}h\beta)$
2. $\mathcal{K}h\alpha \to \mathcal{K}h(\beta \to \alpha)$
3. $\mathcal{K}h(\alpha \to (\beta \to \gamma)) \to \mathcal{K}h((\alpha \to \beta) \to (\alpha \to \gamma))$
4. $\mathcal{K}h(\alpha \wedge \beta) \leftrightarrow (\mathcal{K}h\alpha \wedge \mathcal{K}h\beta)$
5. $\mathcal{K}h\alpha \to \mathcal{K}h(\beta \to (\alpha \wedge \beta))$
6. $\mathcal{K}h(\alpha \vee \beta) \leftrightarrow (\mathcal{K}h\alpha \vee \mathcal{K}h\beta)$
7. $\mathcal{K}h(\alpha \to \beta) \wedge \mathcal{K}h(\beta \to \gamma) \to \mathcal{K}h((\alpha \vee \beta) \to \gamma)$
8. $\mathcal{K}h(\bot \to \alpha)$

Rules:

1. if $\mathcal{K}h\alpha \to \mathcal{K}h\beta$ is provable then $\mathcal{K}h(\alpha \to \beta)$ is provable
2. Modus Ponens

We can show $\vdash \mathcal{K}h\alpha$ iff $\vDash \mathcal{K}h\alpha$ iff $\vdash_{\mathbf{IL}} \alpha$.

On the other hand, axiomatizing the full dynamic epistemic logic in the original setting of Medvedev is still hard (ongoing work with Yunsong Wang), but we can handle various other intermediate logics neatly.

Example: Inquisitive Logic & More

Joint works with Haoyu Wang and Yunsong Wang

- Haoyu Wang, Yanjing Wang, Yunsong Wang: Inquisitive Logic as an Epistemic Logic of Knowing How. Annals of Pure and Applied Logic 173(10), 103145 2022
- Haoyu Wang, Yanjing Wang, Yunsong Wang: An Epistemic Interpretation of Tensor Disjunction. Advances in Modal Logic 2022

Support semantics for Propositional Inquisitive Logic

Definition (Support)

Given **P** and an information model $\mathcal{M} = \langle W, V \rangle$, an *(information) state* $s \subseteq W$ is a subset of W. *Support* is a relation between states and formulas (written as $\mathcal{M}, s \Vdash \alpha$):

1. $\mathcal{M}, s \Vdash p$ iff $\forall w \in s, p \in w$.
2. $\mathcal{M}, s \Vdash \bot$ iff $s = \varnothing$.
3. $\mathcal{M}, s \Vdash (\alpha \wedge \beta)$ iff $\mathcal{M}, s \Vdash \alpha$ and $\mathcal{M}, s \Vdash \beta$.
4. $\mathcal{M}, s \Vdash (\alpha \vee \beta)$ iff $\mathcal{M}, s \Vdash \alpha$ or $\mathcal{M}, s \Vdash \beta$.
5. $\mathcal{M}, s \Vdash (\alpha \to \beta)$ iff $\forall t \subseteq s$: if $\mathcal{M}, t \Vdash \alpha$ then $\mathcal{M}, t \Vdash \beta$.

Inquisitive logic, **InqB**, is the set of **PL**-formulas that are valid in inquisitive semantics, i.e. the set of formulas that are supported by all states.

Axioms

`INTU`	Intuitionistic validities
`DN`	$\neg\neg p \to p$ for all $p \in \mathbf{P}$

Rules:

`MP` $\dfrac{\alpha, \alpha \to \beta}{\beta}$

with *one of* the following axiom schemata:

`KP` $(\neg\alpha \to \beta \vee \gamma) \to (\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma)$

$\mathtt{ND}_k$ $(\neg\alpha \to \bigvee_{1 \leq i \leq k} \neg\beta_i) \to \bigvee_{1 \leq i \leq k}(\neg\alpha \to \neg\beta_i)$

Uniform substitution is not valid thus **InqB** is a *weak* intermediate logic.

Apply our proposal [Wang Wang& Wang APAL 2022]

$$\textbf{Inquisitive truth of } \alpha = \text{knowing how to resolve } \alpha$$
$$= \text{knowing how } \alpha \text{ is true}$$

Similar idea appeared as early as in Ciardelli (2009).

The distinct difference is that for atomic propositions we have special assignments:

$$j_1(p) = \{p\}$$

Thus $j_2(p) = \{p\}$ or $j_2(p) = \emptyset$, i.e., p has at most one resolution.

In this case, the model can be simplified as standard S5 model $\langle W, \sim, V\rangle$ where $w \in V(p)$ means p has a (unique) resolution on w, and $\sim$ is **total.**

With the same formalized BHK definitions of resolutions on each world, we can show $\{\alpha \mid \mathcal{K}h\alpha \text{ is valid}\}$ is exactly **InqB** (valid formulas in the standard propositional inquisitive logic).

Crucial axiom

The assumption that the atomic p has at most one resolution make the following formula valid:

$$\mathcal{K}p \to \mathcal{K}hp$$

Note that since $\mathcal{K}h\alpha \to \mathcal{K}\alpha$ is valid as before, $\mathcal{K}p \leftrightarrow \mathcal{K}hp$ is valid.

Distinctive feature of **InqB**: $\neg\neg p \to p$ (for atomic p)

$$\mathcal{K}h(\neg\neg p \to p)$$

$$\mathcal{K}\Box(\mathcal{K}h\neg\neg p \to \mathcal{K}hp)$$

$$\mathcal{K}\Box(\mathcal{K}p \to \mathcal{K}hp)$$

We gave a complete axiomatization of the logic with $\mathcal{K}h, \mathcal{K}, \Box$, and show that it has exactly the expressive power as S5 modal logic. Many known results about **InqB** becomes transparent.

Complete axiomatization

System SDELKh

Axioms

TAUT	Propositional tautologies	KhK	$\mathcal{K}h\alpha \to \mathcal{K}\alpha$
DISTK	$\mathcal{K}(\varphi \to \psi) \to (\mathcal{K}\varphi \to \mathcal{K}\psi)$	KKhp	$\mathcal{K}p \leftrightarrow \mathcal{K}hp$
T	$\mathcal{K}\varphi \to \varphi$	Kh$\bot$	$\mathcal{K}h\bot \leftrightarrow \bot$
4	$\mathcal{K}\varphi \to \mathcal{K}\mathcal{K}\varphi$	Kh$\vee$	$\mathcal{K}h(\alpha \vee \beta) \leftrightarrow \mathcal{K}h\alpha \vee \mathcal{K}h\beta$
5	$\neg\mathcal{K}\varphi \to \mathcal{K}\neg\mathcal{K}\varphi$	Kh$\wedge$	$\mathcal{K}h(\alpha \wedge \beta) \leftrightarrow \mathcal{K}h\alpha \wedge \mathcal{K}h\beta$
DIST$_\Box$	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	Kh$\to$	$\mathcal{K}h(\alpha \to \beta) \leftrightarrow \mathcal{K}\Box(\mathcal{K}h\alpha \to \mathcal{K}h\beta)$
T$_\Box$	$\Box\varphi \to \varphi$	$4_{\mathtt{Kh}}$	$\mathcal{K}h\alpha \to \mathcal{K}\mathcal{K}h\alpha$
4$_\Box$	$\Box\varphi \to \Box\Box\varphi$	$5_{\mathtt{Kh}}$	$\neg\mathcal{K}h\alpha \to \mathcal{K}\neg\mathcal{K}h\alpha$
PR	$\mathcal{K}\Box\varphi \to \Box\mathcal{K}\varphi$	EU$_k$	$\alpha \wedge \bigwedge_{1\le i\le k}\widehat{\mathcal{K}}(\alpha \wedge \alpha_i) \to \Diamond(\mathcal{K}\alpha \wedge \bigwedge_{1\le i\le k}\widehat{\mathcal{K}}\alpha_i)$
Per	$\alpha \to \Box\alpha$		$(k \in \mathbb{N}, \alpha_i \in$ **PL** for $i \in \mathbb{N})$
Ver	$\alpha \to \Diamond\mathcal{K}h\alpha$		

where $\alpha \in$ **PL**, $p \in$ **P**, $\varphi \in$ **DELKh**

Rules:

MP $\dfrac{\varphi, \varphi \to \psi}{\psi}$ NECK $\dfrac{\vdash \varphi}{\vdash \mathcal{K}\varphi}$ NEC$_\Box$ $\dfrac{\vdash \varphi}{\vdash \Box\varphi}$

Validities

The following schemata are provable in SDELKh, where $\alpha \in \mathbf{PL}$ and $\varphi \in \mathbf{DEL}$ (i.e., $\mathcal{K}h$-free).

$$\texttt{INV} \quad \Box\alpha \leftrightarrow \alpha \tag{1}$$

$$\texttt{KINV} \quad \Box\mathcal{K}\alpha \leftrightarrow \mathcal{K}\alpha \tag{2}$$

$$\texttt{hKINV} \quad \Box\widehat{\mathcal{K}}\alpha \leftrightarrow \alpha \tag{3}$$

$$\texttt{B}\vee \quad \Box(\alpha \vee \varphi) \leftrightarrow \alpha \vee \Box\varphi \tag{4}$$

$$\texttt{BK}\vee \quad \Box(\widehat{\mathcal{K}}\alpha \vee \mathcal{K}\alpha_1 \vee \cdots \vee \mathcal{K}\alpha_n) \leftrightarrow \alpha \vee \mathcal{K}(\alpha \vee \alpha_1) \vee \cdots \vee \mathcal{K}(\alpha \vee \alpha_n) \tag{5}$$

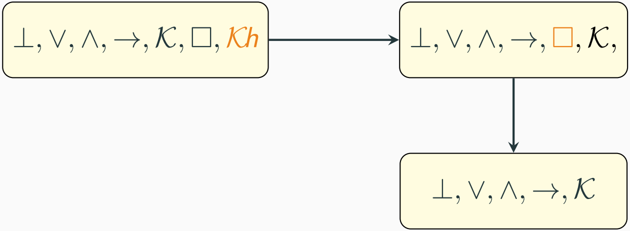
By using these formulas we can eliminate the $\Box$ if there are only $\Box$ and $\mathcal{K}$ cf. techniques in [Balbiani et al. 2008]

Completeness

Theorem (Completeness)

System SDELKh *is a complete axiomatization of* **InqKhL** *over S5 models.*

Proof By a reduction technique:



■

Qua expressivity, inquisitive logic (viewed as the $\mathcal{K}h$ fragment in our language) is a fragment of the standard epistemic logic. This coincides the earlier result by Ciardelli (2018).

Old concepts in the new light

Let $\mathcal{M}_s$ be the corresponding epistemic model of the state s.

Inquisitive semantics	Our epistemic semantics
Information model	single-agent S5 epistemic model
non-empty states	epistemic submodels
support $(\mathcal{M}, s \Vdash \alpha)$	know-how $(\mathcal{M}_s \vDash \mathcal{K}h\alpha)$
alternatives for α in $\mathcal{M}$	maximal submodels of $\mathcal{M}$ satisfying $\mathcal{K}h\alpha$
proposition expressed by α in $\mathcal{M}$	set of submodels of $\mathcal{M}$ for $\mathcal{K}h\alpha$
α is inquisitive in $\mathcal{M}$	there are two maximal submodels satisfying $\mathcal{K}h\alpha$
α is informative in $\mathcal{M}$	a world not in max. submodels of $\mathcal{M}$ for $\mathcal{K}h\alpha$

Characterization

We can define the relative notions of assertions and questions.

Questions and assertions

- α is a *question* in s iff it is not informative in s;
- α is an *assertion* in s iff it is not inquisitive in s.

Proposition (Informativeness and inquisitiveness)

- α is informative in $s \neq \varnothing$ iff $\mathcal{M}_s \vDash \neg\mathcal{K}\alpha$. Thus α is a question in $s \neq \varnothing$ iff $\mathcal{M}_s \vDash \mathcal{K}\alpha$;
- α is inquisitive in $s \neq \varnothing$ iff $\mathcal{M}_s \vDash \widehat{\mathcal{K}}\Diamond(\mathcal{K}\alpha \wedge \neg\mathcal{K}h\alpha)$. Thus α is an assertion in s iff $\mathcal{M}_s \vDash \mathcal{K}\Box(\mathcal{K}\alpha \rightarrow \mathcal{K}h\alpha)$.

Inquisitive logic with tensor [AiML 2022]

We look at (propositional) dependence logic closely related to inquisitive logic [Yang 2016]. The distinct tensor disjunction $\otimes$ there is intricate without an intuitive interpretation.

We add the tensor disjunction $\otimes$ to inquisitive logic and repeat the story, but with a stronger language involving propositional quantifiers to decode the tensor.

Complete axiomatization is also obtained.

It turns out that the tensor disjunction is essentially the *weak disjunction* discussed by Medvedev!

We generalized the binary tensor with parameters k, n capture the epistemic situation that you did an exam of n questions and you know at least k out of your n answers are correct.

Conclusions

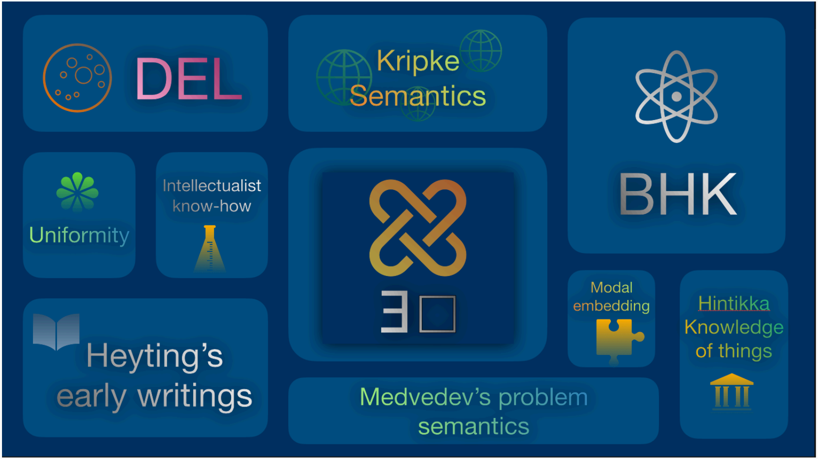
Conclusions

- The "truth" in Intuitionistic and intermediate logic can be viewed as knowing how to prove/solve/resolve.
- Intuitionistic reasoning is about preserving know-how
- We use dynamic epistemic logics of knowing how to decode intuitionistic and intermediate logics
- BHK and Kripke semantics are unified as Kripke models are abstraction of the temporal unravelling
- We can have intuitive understanding of known results and prehaps new insight for new results
- Classical reasoning can be mixed with the intuitionistic reasoning, we don't need to choose side.

Making the implicit explicit by diving into “why”

and we met some of the greatest minds

The ingredients



Building bridges instead of walls

- Classical vs. non-calssical
- World vs. states
- Non-modal vs. epistemic
- *De dicto* vs. *de re*
- Quantifier vs. modaliy

You do not need to take side.

Points to make

- A bit of formal philosophy and linguistics bring you further
- Suitable language can make the implicit explicit
- Modalities bring things from meta language to object language
- Bundled modalities help you see more things
- Concepts helps you to understand intuitively.
- Be careful when you combine intuitionistic or intermediate logics with other modalities.

Bundles in Non-classical Logics

Knowing How to Understand Intuitionistic Logic

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June 25th, 2025

I **think** I know how to understand intuitionistic logic, but I can be wrong...

Knowing How to Understand Intuitionistic Logic

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