Bundles in Deontic Logic

Bundle+BHK for non-normal modal logic

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NASSLLI25, UW

June 26th, 2025

Background

Phenomena in natural language as "Icebergs"

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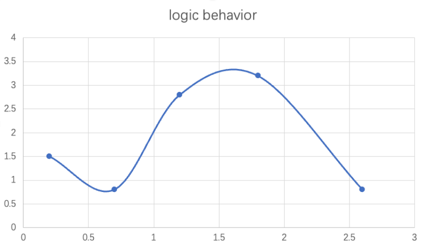
What shall we do when seeing an iceberg?

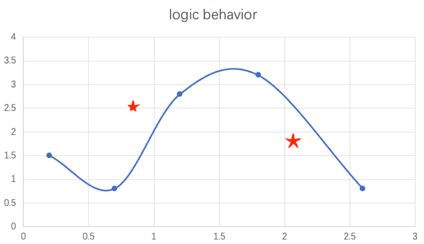
Using various logic techniques to fit the "data"...



The problem of overfitting

New phenomena by looking at the iceberg from other angles ...

Using logic techniques to fit the new "data"... When is the end?



Data Fitting vs. Understanding Why

Deontic Logic: ocean with lots of icebergs...

There are many logical puzzles in Standard Deontic Logic (SDL), deviating from normal modal logic when taking Obligation ($\mathbf{O}$) as a $\Box$ and Permission ($\mathbf{P}$) as a $\Diamond$.

Among many others:

- Ross' paradox: $\mathbf{O}p \rightarrow \mathbf{O}(p \vee q)$ and $\mathbf{P}p \rightarrow \mathbf{P}(p \vee q)$ are intuitively invalid, but valid in SDL.
- Free choice: $\mathbf{P}(p \vee q) \rightarrow \mathbf{P}p \wedge \mathbf{P}q$ is intuitively valid, but logically invalid in SDL.

We focus on *Strong* Permission ($\mathbf{P}$), the permissions *explicitly granted* rather than simply not being forbidden. Strong permissions exhibit the property of free choice (FCP).

Basic observation and questions

Deontic modalities may be more than what they appear to be!

- Could they also be *bundles* of a quantifier and a usual modality?
- If so, quantifying over what?

Formulas inside deontic modalities might not be propositions

- Then what are they?
- How can we treat them formally?

Further observations regarding quantifiers and bundles

If a hidden quantifier were present, what would it quantify?

- The distinction between action types and tokens, i.e., individual actions (well-known in the literature)
- Deontic sentences mention only action types
- But the semantics may be about tokens of those types

What could be the bundle for strong permission **P**?

- It is clearly not $\exists x \Box$, but it might be $\forall x \Diamond$ (Hintikka 1971).
- $\mathbf{P}\alpha$: each token of action type α is executable on some deontically ideal successor of the current world.
- If you are permitted to take one day off next week. Each relevant token (taking Monday off, taking Tuesday off...) should be executable on some ideal world.

Further observations: formalizing action type and token

- Propositional formulas as action types
- They do not have truth values, though can be assigned!
- They can be viewed as collections of action tokens
- We borrow the BHK-like formalism to capture them

Brouwer-Heyting-Kolmogorov (BHK) interpretation

BHK *proof* interpretation of *connectives*:

(H1) A proof of $\alpha \wedge \beta$ is given by presenting a proof of α and a proof of β

(H2) A proof of $\alpha \vee \beta$ is given by presenting either a proof of α or a proof of β

(H3) A proof of $\alpha \to \beta$ is a construction which *transforms* any proof of α into some proof of β

(H4) Absurdity $\bot$ has *no* proof.

$\neg\alpha$ is the abbriviation of $\alpha \to \bot$.

We can define the relation between action tokens and types recursively like the above.

Further observations: formalizing action type and token

- Propositional formulas as action types
- They do not have truth values, though can be assigned!
- They can be viewed as a collections of action tokens
- We borrow the BHK-like formalism to capture them

	Intuitionistic Logic	Deontic Logic
prop. formulas	type of problems	type of actions
token	solution/proof	individual act
modality	know-how	permission
bundle	$\exists\Box$	$\forall\Diamond$

You will be rewarded if you get the semantics (more or less) right.

[Wang&Wang DEON23]: predicting new linguistic phenomena

Valid in our framework			
FC	$\mathbf{P}(\alpha \vee \beta) \leftrightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$	CD	$\mathbf{P}(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow \mathbf{P}((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
CE	$\mathbf{P}(\alpha \wedge \beta) \rightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$	DCl	$\mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \rightarrow \mathbf{P}(\alpha \vee (\beta \wedge \gamma))$
Invalid in our framework			
CA	$(\mathbf{P}\alpha \wedge \mathbf{P}\beta) \rightarrow \mathbf{P}(\alpha \wedge \beta)$	DCr	$\mathbf{P}(\alpha \vee (\beta \wedge \gamma)) \rightarrow \mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$
RP	$\mathbf{P}\alpha \rightarrow \mathbf{P}(\alpha \vee \beta)$	EX	$\mathbf{P}\alpha \rightarrow \mathbf{P}(\alpha \wedge \alpha)$

`DCr` is invalid: imagine you are given a coupon that permits you to take a hamburger *or* a menu of French fries *and* salad, this does not mean you can take a hamburger *or* fries, *and* a hamburger *or* salad.

`CA` and `DCr` are valid in Boolean-algebra-based approaches, such as [3, 4]; `CD` is invalid in [2]; `DCr` is valid in the hybrid approach based on BSML [1]; and `CE` is not valid in [5].

About the "innocent" EX: $\mathbf{P}\alpha \to \mathbf{P}(\alpha \land \alpha)$

It is not as innocent as it seems. Under free choice and acceptable distribution, it leads to the unacceptable $\mathbf{P}(\alpha \lor \beta) \to \mathbf{P}(\alpha \land \beta)$!

$$\begin{array}{lll}
 & \mathbf{P}(\alpha \lor \beta) & \\
\Longrightarrow & \mathbf{P}((\alpha \lor \beta) \land (\alpha \lor \beta)) & (\mathtt{EX}) \\
\Longleftrightarrow & \mathbf{P}(((\alpha \lor \beta) \land \alpha) \lor ((\alpha \lor \beta) \land \beta)) & (\mathtt{CD}) \\
\Longleftrightarrow & \mathbf{P}((\alpha \lor \beta) \land \alpha) \land \mathbf{P}((\alpha \lor \beta) \land \beta) & (\mathtt{FC}) \\
\Longleftrightarrow & \mathbf{P}((\alpha \land \alpha) \lor (\alpha \land \beta)) \land \mathbf{P}((\beta \land \alpha) \lor (\beta \land \beta)) & (\mathtt{CD}, \text{commutativity}) \\
\Longleftrightarrow & \mathbf{P}(\alpha \land \alpha) \land \mathbf{P}(\alpha \land \beta) \land \mathbf{P}(\beta \land \alpha) \land \mathbf{P}(\beta \land \beta) & (\mathtt{FC}) \\
\Longrightarrow & \mathbf{P}(\alpha \land \beta) & (\mathtt{TAUT})
\end{array}$$

Let's get to the details.

Language and Semantics

Language AT of action types

Definition (Action Type AT)

Given a countable set of propositional letters P, the language of action types (**AT**) is defined as follows (no implication for now):

$$\alpha := p \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha)$$

where $p \in P$.

We use atomic propositional letters to represent atomic action types like "drink coffee", "do homework", "go to hospital", etc. Complex action types like 'eat cookies and drink coffee", "do homework or play computer games" can also be expressed.

Language DLSP of Deontic Logic of Strong Permission

Definition (Language DLSP)

Given **AT**, the language of deontic logic for strong permission (**DLSP**) is defined as follows:

$$\varphi := \bot \mid p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid \neg\varphi \mid \mathbf{P}\alpha$$

where $p \in P$ and $\alpha \in \mathbf{AT}$.

We call formulas containing the deontic operator **P** *deontic formulas* and other formulas *non-deontic*.

Action space

Following the BHK-style definition:

Definition (Action Token Space)

Given P and a non-empty set I of atomic action tokens such that $I \cap \{0,1\} = \emptyset$, an action (token) space S is a function based on I and **AT** satisfying the following constraints:

1. $S(p) \neq \emptyset \subseteq I$ for any $p \in P$;
2. $S(\alpha \wedge \beta) = S(\alpha) \times S(\beta)$;
3. $S(\alpha \vee \beta) = (S(\alpha) \times \{0\}) \cup (S(\beta) \times \{1\})$.

S is a singleton action space if $|S(p)| = 1$ for all $p \in P$. People may treat types and tokens alike for atomic actions.

For example, action tokens for a disjunctive action type $(\alpha \vee \beta)$ are the *disjoint* union of tokens of α and β.

Deontic Model

Definition (Deontic Model)

A deontic model $\mathcal{M}$ for **DLSP** is a tuple (S, W, R, A) where S is an action space, W is a non-empty set of possible worlds, $R \subseteq W \times W$, and A is a binary function over $\mathbf{AT} \times W$ such that for any $p \in P$, $\alpha, \beta \in \mathbf{AT}$ and $w \in W$:

- $A(p, w) \subseteq S(p)$;
- $A(\alpha \wedge \beta, w) = A(\alpha, w) \times A(\beta, w)$;
- $A(\alpha \vee \beta, w) = (A(\alpha, w) \times \{0\}) \cup (A(\beta, w) \times \{1\})$;

A pointed model is a pair $(\mathcal{M}, w)$ where w is in $\mathcal{M}$. A singleton deontic model is a model based on a singleton action space.

The function A gives each deontially good world its *executed* action tokens.

Semantics

Definition (Semantics)

For any $\varphi \in \mathbf{DLSP}$ and any pointed deontic model $\mathcal{M}, w$ where $\mathcal{M} = (S, W, R, A)$, the satisfaction relation is defined as follows:

$\mathcal{M}, w \not\models \bot$		
$\mathcal{M}, w \models p$	$\iff$	$A(p, w) \neq \emptyset$
$\mathcal{M}, w \models (\varphi \wedge \psi)$	$\iff$	$\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models (\varphi \vee \psi)$	$\iff$	$\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models (\varphi \rightarrow \psi)$	$\iff$	$\mathcal{M}, w \not\models \varphi$ or $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models \neg\varphi$	$\iff$	$\mathcal{M}, w \not\models \varphi$
$\mathcal{M}, w \models \mathbf{P}\alpha$	$\iff$	for any $a \in S(\alpha)$, there is a v s.t. wRv and $a \in A(\alpha, v)$

We use $\models_s$ to denote semantic consequence w.r.t. singleton deontic models. We say φ is valid (s-valid) if $\models \varphi$ ($\models_s \varphi$).

Recall the preview

Valid in our framework			
FC	$\mathbf{P}(\alpha \vee \beta) \leftrightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$	CD	$\mathbf{P}(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow \mathbf{P}((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
CE	$\mathbf{P}(\alpha \wedge \beta) \rightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$	DCl	$\mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \rightarrow \mathbf{P}(\alpha \vee (\beta \wedge \gamma))$
Invalid in our framework (without further constraints)			
CA	$(\mathbf{P}\alpha \wedge \mathbf{P}\beta) \rightarrow \mathbf{P}(\alpha \wedge \beta)$	DCr	$\mathbf{P}(\alpha \vee (\beta \wedge \gamma)) \rightarrow \mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$
RP	$\mathbf{P}\alpha \rightarrow \mathbf{P}(\alpha \vee \beta)$	EX	$\mathbf{P}\alpha \rightarrow \mathbf{P}(\alpha \wedge \alpha)$

The commutativity and associativity are valid.

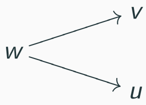
$$\mathbf{P}(\alpha \wedge \beta) \leftrightarrow \mathbf{P}(\beta \wedge \alpha) \qquad \mathbf{P}((\alpha \wedge \beta) \wedge \gamma) \leftrightarrow \mathbf{P}(\alpha \wedge (\beta \wedge \gamma))$$

$$\mathbf{P}(\alpha \vee \beta) \leftrightarrow \mathbf{P}(\beta \vee \alpha) \qquad \mathbf{P}((\alpha \vee \beta) \vee \gamma) \leftrightarrow \mathbf{P}(\alpha \vee (\beta \vee \gamma))$$

Invalidity of DCr: $\mathbf{P}(\alpha \vee (\beta \wedge \gamma)) \rightarrow \mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$

The rightmost part below demonstrates the definition of A on u, v, e.g., $A(p, v) = \{a\}$ and $A(q, u) = \{b\}$.

$S(p) = \{a\}, S(q) = \{b\}, S(r) = \{c\}$



v: $p : \{a\}, r : \emptyset, q : \emptyset$

u: $p : \emptyset, q : \{b\}, r : \{c\}$

$S(p \vee (q \wedge r))$ contains $(a, 0)$ and $((b, c), 1)$ only, which are executable on v and u respectively, thus $\mathbf{P}(p \vee (q \wedge r))$ is true on w. However, the token $((a, 0), (c, 1))$ in $S((p \vee q) \wedge (p \vee r))$ is not executable on u nor v, thus $\mathbf{P}((p \vee q) \wedge (p \vee r))$ is false on w. Note that this model is also a singleton model so DCr is not s-valid.

A weaker version of EX over singleton models

The following formula (denoted by EXP) is valid with respect to the class of singleton deontic models:

$$\vDash_s \mathbf{P}(p_1 \wedge \ldots \wedge p_k) \rightarrow \mathbf{P}(m_1 \cdot p_1 \wedge \ldots \wedge m_k \cdot p_k),$$

where $p_1, \ldots p_k \in P$ are pairwise distinct, $k, m_i \in \mathbb{N}_{>0}$ for any $1 \leq i \leq k$. Here $m_i \cdot p_i$ represents the conjunction of m_i copies of p_i.

Proof Systems

Proof Systems (no replacement of equals in P)

System $\mathbb{DLSP}$

Axioms	
$(\texttt{TAUT})$	Propositional Tautologies
$(\texttt{FC})$	$\mathbf{P}(\alpha \vee \beta) \leftrightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$
$(\texttt{CE})$	$\mathbf{P}(\alpha \wedge \beta) \rightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$
$(\texttt{COM}_\wedge)$	$\mathbf{P}(\alpha \wedge \beta) \leftrightarrow \mathbf{P}(\beta \wedge \alpha)$
$(\texttt{ASSO}_\wedge)$	$\mathbf{P}((\alpha \wedge \beta) \wedge \gamma) \leftrightarrow \mathbf{P}(\alpha \wedge (\beta \wedge \gamma))$
$(\texttt{CD})$	$\mathbf{P}(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow \mathbf{P}((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
Rules	
$(\texttt{MP})$	Given φ and $(\varphi \rightarrow \psi)$, infer ψ.

System $\mathbb{DLSP}^s$

	System $\mathbb{DLSP}$ with the following axiom
$(\texttt{EXP})$	$\mathbf{P}(p_1 \wedge \ldots \wedge p_k) \rightarrow \mathbf{P}(m_1 \cdot p_1 \wedge \ldots \wedge m_k \cdot p_k)$

Normal form

We use $\mathbb{DLSP}$ to rewrite a **DLSP**-formula into a conjunction of formulas in the shape of $\mathbf{P}(p_1 \wedge ... \wedge p_n)$.

$$\mathbf{P}(p_1 \vee (p_2 \wedge ((p_3 \vee p_4) \wedge p_5))).$$

The formula is logically equivalent to

1. $\mathbf{P}p_1 \wedge \mathbf{P}(p_2 \wedge ((p_3 \vee p_4) \wedge p_5))$ $\quad(\texttt{FC})$
2. $\mathbf{P}p_1 \wedge \mathbf{P}((p_5 \wedge p_2) \wedge (p_3 \vee p_4))$ $\quad(\texttt{ASSO}_\wedge + \texttt{COM}_\wedge)$
3. $\mathbf{P}p_1 \wedge \mathbf{P}(((p_5 \wedge p_2) \wedge p_3) \vee ((p_5 \wedge p_2) \wedge p_4))$ $\quad(\texttt{CD})$
4. $\mathbf{P}p_1 \wedge \mathbf{P}((p_5 \wedge p_2) \wedge p_3) \wedge \mathbf{P}((p_5 \wedge p_2) \wedge p_4)$ $\quad(\texttt{FC})$

Lemma (Normal Form for $\mathbf{P}\alpha$)

*For any $\alpha \in$ **AT**, $\mathbf{P}\alpha$ is logically equivalent to a formula of the form $(\mathbf{P}\beta_1 \wedge ... \wedge \mathbf{P}\beta_k)$ where for each $1 \leq i \leq k$, β_i is in the shape of $\mathbf{P}(p_1 \wedge ... \wedge p_n)$, which is called a **normal form** for $\mathbf{P}\alpha$.*

Normal form

For any formula $\varphi \in \mathbf{DLSP}$, φ is logically equivalent to a formula in the following language (denoted by $\mathbf{DLSP}^*$):

$$\psi ::= \bot \mid p \mid \mathbf{P}(p_1 \wedge ... \wedge p_n) \mid \neg\psi \mid (\psi \wedge \psi) \mid (\psi \vee \psi) \mid (\psi \rightarrow \psi),$$

where $p, p_1, ..., p_n \in P$.

To show the completeness, we will construct for each consistent set of formulas a model.

Completeness

All-distinct action token

Note that due to the validity of $\texttt{ASSO}_\wedge$ and $\texttt{COM}_\wedge$, we will treat an action token of type $(p_1 \wedge \ldots \wedge p_n)$ as an n-ary tuple of action tokens modulo paring.

Definition (All-Distinct Token)

An action token of type $(p_1 \wedge \ldots \wedge p_n)$ is *all-distinct* if tokens of the same atomic action type in the tuple are pairwise distinct.

We need to first build the action spaces before constructing the canonical model.

Canonical action space

Now let Σ be a maximally $\mathbb{DLSP}$-consistent set of **DLSP*** formulas.

Definition (Canonical Action Space)

Given Σ, we define S^C_Σ by distinguishing the two cases of $p \in P$:

- If there is an $i \in \mathbb{N}_{>0}$ such that the formula $\neg\mathbf{P}(i \cdot p) \in \Sigma$, assume that n is the least and let $S^C_\Sigma(p) := \{p^1, p^2, ..., p^n\}$, in which each p^i is the propositional letter p superscript with the numeral i.
- If not, i.e., $\mathbf{P}(i \cdot p) \in \Sigma$ for all $i \in \mathbb{N}_{>0}$, let $S^C_\Sigma(p) := \{p^1, p^2, \ldots\}$.

For any composite $\alpha \in \mathbf{AT}$, we define $S^C_\Sigma(\alpha)$ recursively as in the definition of S.

Note that for distinct $p, q \in P$, $S^C_\Sigma(p) \cap S^C_\Sigma(q) = \emptyset$.

Existence of all-distinct token

Lemma

For any formula φ of the form $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k})$ where p_{t_i}, p_{t_j} are pairwise distinct, if $\varphi \in \Sigma$, then for any $1 \leq j \leq k$, $m_j < |S^{C}_{\Sigma}(p_{t_j})|$.

Proof.

Prove by contradiction. □

This lemma shows the size of the action space is *more than enough* to guarantee the existence of all-distinct action tokens of the type $(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k})$ when $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k}) \in \Sigma$.

The very idea of canonical model

Based on the lemma and S^C_Σ, we will build a pointed deontic model $\mathcal{M}^C_\Sigma, w$ such that the truth lemma holds.

The idea is simple: given a designated world w, build the accessible worlds according to formulas $\mathbf{P}(m_1 \cdot p_{t_1} \wedge ... \wedge m_k \cdot p_{t_k}) \in \Sigma$.

The subtlety is that we should only realize action tokens that are *necessary* to witness the truth of those φ, but no more, for we also need tokens not realizable to witness $\neg\mathbf{P}(p_1 \wedge ... \wedge p_n) \in \Sigma$. The later task is doable because we have some spare tokens in S^C_Σ based on the previous Lemma.

Functional representation

Fixing an ordering of propositional letters $p_0, p_1, p_2, \ldots$, we only need to consider $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k}) \in \Sigma$ such that p_{t_i} and p_{t_j} are distinct and ordered, e.g., $\mathbf{P}(3 \cdot p_2 \wedge 4 \cdot p_6)$.

Definition

For any φ of the form $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k}) \in \Sigma$ such that p_{t_i} and p_{t_j} are distinct and ordered according to the order of propositional letters, we define $f_\varphi : \mathbb{N} \to \mathbb{N}$ such that

$$f_\varphi(i) = \begin{cases} m_j & i = t_j \text{ for some } 1 \leq j \leq k; \\ 0 & i \neq t_j \text{ for any } 1 \leq j \leq k. \end{cases}$$

For example, $\mathbf{P}(3 \cdot p_2 \wedge 4 \cdot p_6)$ is represented by the function f such that $f(2) = 3$, $f(6) = 4$ and $f(i) = 0$ for any $i \in \mathbb{N} \setminus \{2, 6\}$. We collect these (countably many) functions in F_Σ.

Functional representation

Definition

For any $f \in F_\Sigma$, we define $G_f := \{g : \mathbb{N} \to \mathcal{P}(\bigcup_{p \in P}(S^C_\Sigma(p))) \mid$ for any $i \in \mathbb{N}, g(i) \subseteq S^C_\Sigma(p_i)$ and $|g(i)| = f(i)\}$.

Intuitively, each $g \in G_f$ assigns a subset of the canonical action space of each p_i whose cardinality is $f(i)$. It follows if $f(i) = 0$ then $g(i) = \emptyset$. In fact, each $g \in G_f$ can be treated as an *all-distinct* token of the type in φ. Let $G_\Sigma = \bigcup\{G_f \mid f \in F_\Sigma\}$.

Proposition

Given a MCS Σ and any distinct $f, f' \in F_\Sigma$, we have: (1) G_f is not empty; (2) $G_f \cap G_{f'} = \emptyset$.

Canonical deontic model

Definition (Canonical Deontic Model)

Given a MCS Σ, we define the model $\mathcal{M}^C_\Sigma = (S^C_\Sigma, W^C, R^C, A^C)$ where:

- $W^C = \{w\} \cup G_\Sigma$; $R^C = \{(w, g) \mid g \in G_\Sigma\}$;
- $A^C(p_i, u) = \begin{cases} S^C_\Sigma(p_i) & \text{if } u = w \text{ and } p_i \in \Sigma, \\ \emptyset & \text{if } u = w \text{ and } p_i \notin \Sigma, \\ u(i) & \text{if } u \in G_\Sigma; \end{cases}$

and $A^C(\alpha, u)$ for composite α is defined as in definition of deontic model.

If $g \in G_\Sigma$ then there is a unique $f \in F_\Sigma$ s.t. $g \in G_f$. Intuitively, each $g \in G_f$ realizes some all-distinct token of the formula $\mathbf{P}\alpha \in \Sigma$ corresponding to f, and G_f realize all the necessary tokens.

Truth Lemma for $\mathbb{DLSP}$

Lemma (Truth Lemma for $\mathbb{DLSP}$)

Given a MCS Σ. For any $\varphi \in \Sigma$,

$$\mathcal{M}^C_\Sigma, w \vDash \varphi \Longleftrightarrow \varphi \in \Sigma.$$

Proof.

Prove by induction on the structure of φ. We only show the inductive case when $\varphi = \mathbf{P}(p_1 \wedge ... \wedge p_n)$. By ($\mathtt{COM}_\wedge$) and ($\mathtt{ASSO}_\wedge$), φ is logically equivalent to a formula ψ of the form $\mathbf{P}(m_1 \cdot p_{t_1} \wedge ... \wedge m_k \cdot p_{t_k})$ where p_{t_i} and p_{t_j} are pairwise distinct and ordered.

Truth Lemma for $\mathbb{DLSP}$

Proof.

$\Leftarrow$: Assume that $\psi \in \Sigma$. We have the corresponding $f_\psi \in F_\Sigma$. Then each all-distinct token of type in ψ is represented and thus realized by some $g \in G_{f_\psi}$. And this will guarantee all tokens be realized.

$\Rightarrow$: Assume that $\psi \notin \Sigma$. To show $\mathcal{M}^C_\Sigma, w \not\models \psi$, we need to find some token in $S^C_\Sigma(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k})$ cannot be witnessed by any successor. The crucial point here is that our definition of S^C_Σ and A^C together guarantee that some action tokens are indeed **left out** at every $g \in G_\Sigma$.

Truth Lemma for $\mathbb{DLSP}$

Proof.

Now we consider two cases:

- If for any $1 \leq j \leq k$, $m_j \leq |S^C_\Sigma(p_{t_j})|$, we take an all-distinct token $x \in S(m_1 \cdot p_{t_1} \wedge ... \wedge m_k \cdot p_{t_k})$ and show it is not realizable in G_Σ, thus $\mathcal{M}^C_\Sigma, w \not\models \psi$. Suppose not, so there is a $g \in G_\Sigma$ that realizes x, then there is a unique f such that $g \in G_f$. Since g realizes all-distinct token x, then we have $f(t_j) = |g(t_j)| = |A^C(p_{t_j}, g)| \geq m_j$ for any $1 \leq j \leq k$. Due to our construction, there must be a $\chi \in \Sigma$ such that $f = f_\chi$. Therefore, χ must be of the form $\mathbf{P}((m'_1 \cdot p_{t_1} \wedge ... \wedge m'_k \cdot p_{t_k}) \wedge (m'_{k+1} \cdot p_{t_{k+1}} \wedge ... \wedge m'_{k+l} \cdot p_{t_{k+l}})) \in \Sigma$ such that $m'_j = f(t_j) \geq m_j$. By (CE) and (MP), $\psi \in \Sigma$, contradicting to the assumption that $\psi \notin \Sigma$.

Proof.

- If there is $1 \leq j \leq k$ such that $m_j > |S^C_\Sigma(p_{t_j})|$, thus $S^C_\Sigma(p_{t_j})$ is finite, say $|S^C_\Sigma(p_{t_j})| = n$. Suppose towards a contradiction that $\mathcal{M}^C_\Sigma, w \vDash \psi$. Thus by the validity of $\mathtt{CE}$, $\mathcal{M}^C_\Sigma, w \vDash \mathbf{P}(n \cdot p_{t_j})$. Hence, to realize the token using all the atomic tokens in $S^C_\Sigma(p_{t_j})$, there must be a $g \in G$ such that $A^C(p_{t_j}, g) = g(t_j) = S^C_\Sigma(p_{t_j})$. Further there must be a unique f such that $g \in G_f$ and $f(t_j) = |g(t_j)| = |S^C_\Sigma(p_{t_j})| = n$. Therefore there is a $\chi \in \Sigma$ such that $f = f_\chi$. However this means χ must be in the shape of $\mathbf{P}(n \cdot p_{t_j} \wedge \beta) \in \Sigma$. By ($\mathtt{CE}$), $\mathbf{P}(n \cdot p_{t_j}) \in \Sigma$ contradicting to the fact that $|S^C_\Sigma(p_{t_j})| = n$. Therefore $\mathcal{M}^C_\Sigma, w \nvDash \psi$.

$\square$

Completeness Theorem for $\mathbb{DLSP}$

Based on the truth lemma, by a Lindenbaum-like argument, we can show:

Theorem (Completeness Theorem for $\mathbb{DLSP}$)

$\mathbb{DLSP}$ *is strongly complete with respect to the class of all deontic models.*

Note that $\mathbb{DLSP}$ is also complete over all *serial* models, i.e., the models where every node has a successor.

Canonical singleton action space

Now we prove completeness theorem for $\mathbb{DLSP}^s$. Let Σ be a maximally $\mathbb{DLSP}^s$-consistent set of **DLSP*** formulas.

Definition (Canonical Singleton Action Space)

Given Σ, we define the canonical singleton action space S^s_Σ such that $S^s_\Sigma(p) := \{p\}$ for any $p \in P$ and $S^s_\Sigma(\alpha)$ is defined recursively as above for any composite $\alpha \in \mathbf{AT}$.

To define the canonical singleton deontic model $\mathcal{M}^s_\Sigma = (S^s_\Sigma, W^s, R^s, A^s)$, still fixing an ordering of propositional letters in advance, we essentially apply the same method as before.

Canonical singleton deontic model

However, our definition will be simplified here. Since $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k})$ is logically equivalent to $\mathbf{P}(p_{t_1} \wedge \ldots \wedge p_{t_k})$ by extra validities ($\texttt{EXP}$) in $\mathbb{DLSP}^s$, we only consider formulas φ of the latter form in Σ and define f_φ, G_{f_φ} as before. So, for any $i \in \mathbb{N}$, if $i = t_j$, then $f_\varphi(i) = 1$, and otherwise $f_\varphi(i) = 0$. And by this feature of f_φ and S^s_Σ as singleton action space, there is indeed a unique $g \in G_{f_\varphi}$ such that $g(i) = \{p_{t_j}\}$ if $i = t_j$ and $g(i) = \emptyset$ otherwise. We collect all such g in G'_Σ.

Canonical singleton deontic model

Definition (Canonical Singleton Deontic Model)

Given Σ, we define the singleton deontic model $\mathcal{M}^s_\Sigma = (S^s_\Sigma, W^s, R^s, A^s)$ where:

- $W^s = \{v\} \cup G'_\Sigma$; $R^s = \{(v, g) \mid g \in G'_\Sigma\}$;
- $A^s(p_i, u) = \begin{cases} S^s_\Sigma(p_i) & \text{if } u = v \text{ and } p_i \in \Sigma, \\ \emptyset & \text{if } u = v \text{ and } p_i \notin \Sigma, \\ u(i) & \text{if } u \in G'_\Sigma; \end{cases}$

and $A^s(\alpha, u)$ for composite α is defined as in Definition of deontic model.

Completeness theorem for $\mathbb{DLSP}^s$

Lemma (Truth Lemma for $\mathbb{DLSP}^s$)

Let Σ be a maximally $\mathbb{DLSP}^s$-consistent set of **DLSP*** *formulas. For any $\varphi \in \Sigma$,*

$$\mathcal{M}^s_\Sigma, v \vDash_s \varphi \Longleftrightarrow \varphi \in \Sigma.$$

Theorem (Completeness Theorem for $\mathbb{DLSP}^s$)

$\mathbb{DLSP}^s$ is strongly complete with respect to the class of all singleton deontic models.

Extensions

Higher-order permission

Giving a permission itself can also be an action type!

Then we can express $\mathbf{PP}p$, $\neg\mathbf{P}(p \vee \mathbf{P}q)$, $\mathbf{PP}p \to \mathbf{P}p, \mathbf{P}p \wedge \neg\mathbf{PP}p \ldots$

We can give the interpretation for $\mathbf{P}\alpha$ as a type.

- $S(\mathbf{P}\alpha) = \{c_\alpha\}$.
- $A(\mathbf{P}\alpha, w) = \begin{cases} \{c_\alpha\} & w \vDash \mathbf{P}\alpha \\ \emptyset & \text{otherwise} \end{cases}$

The system $\mathbb{DLSP}$ with the following rule is sound and complete:

$$\text{Given } \vdash \bigwedge \overline{\mathbf{P}\alpha} \to \bigwedge \overline{\mathbf{P}\beta}, \text{ infer } \vdash \chi \to \chi[\bigwedge \overline{\mathbf{P}\beta} / \bigwedge \overline{\mathbf{P}\alpha}].$$

With an extra axiom $\mathbf{PP}\alpha \to \mathbf{P}\alpha$, the logic is complete over transitive frames.

Simultaneous conjunction

In $\mathbb{DLSP}$, $(\mathtt{CE}) : \mathbf{P}(\alpha \wedge \beta) \rightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$ is valid. On the other hand, it also seems controversial, e.g. [2]:

Flight safety: in case of emergency, you are permitted to wear the parachute and jump out of the plane; but you are not permitted to jump directly.

- Here, the conjunctive action is *simultaneous* in the sense that both conjuncts need to be *uniformly* executed by one and the same token.
- For example, sometimes you are required to take different medicines together to guarantee they all work.
- We can express such simultaneous action types by alternatively interpreting conjunction as *intersection* rather than product.

Simultaneous action space and model

Given P, we still work in language $\mathbf{AT}_P$ and $\mathbf{DLSP}_P$.

Definition (Simultaneous Action Token Space)

Given a non-empty set I of atomic action tokens, a simultaneous action (token) space S is a function with domain $\mathbf{AT}_P$ such that:

1. $S(p) \neq \emptyset \subseteq I$ for any $p \in P$;
2. $S(\alpha \vee \beta) = S(\alpha) \cup S(\beta)$ and $S(\alpha \wedge \beta) = S(\alpha) \cap S(\beta)$.

Definition (Simultaneous Deontic Model)

A simultaneous deontic model $\mathcal{M}$ is a tuple (S, W, R, A) where S is a simultaneous action space, W, R are as before, A matches S and satisfies coinstantiation that for any $\alpha, \beta \in \mathbf{AT}$ and $w \in W$,

$$\text{if } x \in S(\alpha) \cap S(\beta), \text{ then } x \in A(\alpha, w) \iff x \in A(\beta, w).$$

System $\mathbb{DLSP}^{sc}$ and completeness

Axioms	
(TAUT)	Propositional Tautologies
(FC)	$\mathbf{P}(\alpha \vee \beta) \leftrightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$
(AB)	$\mathbf{P}(\alpha \wedge \alpha) \to \mathbf{P}\alpha$
(CI)	$\mathbf{P}\alpha \to \mathbf{P}(\alpha \wedge \beta)$
$(\mathtt{COM}_\wedge)$	$\mathbf{P}(\alpha \wedge \beta) \leftrightarrow \mathbf{P}(\beta \wedge \alpha)$
$(\mathtt{ASSO}_\wedge)$	$\mathbf{P}((\alpha \wedge \beta) \wedge \gamma) \leftrightarrow \mathbf{P}(\alpha \wedge (\beta \wedge \gamma))$
(CD)	$\mathbf{P}(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow \mathbf{P}((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
Rules	
(MP)	Given φ and $(\varphi \to \psi)$, infer ψ.

Theorem (Soundness and Completeness Theorem for $\mathbb{DLSP}^{sc}$)

$\mathbb{DLSP}^{sc}$ *is sound and strongly complete with respect to the class of all simultaneous deontic models.*

Negated action types

Two kinds of action negation

Intuitively, we have (at least) two kinds of action negation.

"Not doing α" means

- either α just being not done.
- or, doing something other than α.

In the scope of strong permission, the former one seems more appropriate:

You are permitted not to do your homework right now,
but you are NOT permitted to play video games.

The permission not to do α allows *just* that α is not done, and nothing more.

Negated action type [Wang&Wang DEON25]

Now we extend our framework with the not-doing action types.

Definition (Action Type $\mathbf{AT}_P$)

Given a countable set P of propositional letters, the language of action types ($\mathbf{AT}_P$) is defined as follows: for $p \in P$,

$$\alpha ::= p \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha) \mid \neg\alpha.$$

Definition (Language $\mathbf{DLSP}_P$)

Given $\mathbf{AT}_p$, the language of deontic logic for strong permission ($\mathbf{DLSP}_P$) is defined as follows: for $p \in P$ and $\alpha \in \mathbf{AT}_P$,

$$\varphi ::= \bot \mid p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \neg\varphi \mid \mathbf{P}\alpha.$$

Intuitively, $\mathbf{P}\neg\alpha$ says it is permitted not to do α. Fix a P below.

Formalizing action negation as not being done

We introduce a unique "negative" token n_α for each $\neg\alpha$ and fix $S(\neg\alpha) = \{n_\alpha\}$: deliberatively not doing α can be viewed as an action.

These tokens are newly added and *not* in our given set I of "positive" atomic action tokens that really can be executed.

Rather, each n_α just acts as a marker semantically indicating the corresponding type is not realized (at a world).

Specifically, at world w, n_α is executed iff all tokens of α are *not* executed, i.e. $A(\neg\alpha, w) \neq \emptyset \iff A(\alpha, w) = \emptyset$. Recall the treatment of negation in inquisitive logic.

Negated action space and deontic model

Definition (Negated Action Space)

Given P, a set I of positive atomic action tokens and a set $N = \{n_\alpha \mid \alpha \in \mathbf{AT}\}$ of negative atomic action tokens such that $I \cap N = \varnothing$, $(I \cup N) \cap \{0, 1\} = \varnothing$, and $n_\alpha \neq n_\beta$ for distinct $\alpha, \beta \in \mathbf{AT}$, a negated action (token) space S is a function with domain $\mathbf{AT}$ such that:

- $S(p), S(\alpha \wedge \beta), S(\alpha \vee \beta)$ *satisfy the constraints as before;*
- $S(\neg\alpha) = \{n_\alpha\}$ *for any* $\alpha \in \mathbf{AT}$.

On the model level, we further require that for any $\alpha \in \mathbf{AT}$,

$$A(\neg\alpha, w) = \begin{cases} \{n_\alpha\} & A(\alpha, w) = \varnothing, \\ \varnothing & A(\alpha, w) \neq \varnothing. \end{cases}$$

Such a definition characterizes our intuition to the negated action $\neg\alpha$ only as α being not done.

Properties preserved

After extension from $\mathbf{AT}^-$ to $\mathbf{AT}$, we still have:

Proposition

For any $\alpha \in \mathbf{AT}$ and pointed deontic model $\mathcal{M}, w$ for **DLSP***,*

1. $S(\alpha) \neq \varnothing$;
2. $A(\alpha, w) \subseteq S(\alpha)$;
3. $\mathcal{M}, w \vDash \alpha \iff A(\alpha, w) \neq \varnothing$.

Note that $S(\alpha)$ is not empty even for intuitively impossible action types. For example, $S(\alpha \wedge \neg\alpha) = \{(a, n_\alpha) \mid a \in S(\alpha)\} \neq \varnothing$ for any α. However, since $A(\neg\alpha, w) \neq \emptyset \iff A(\alpha, w) = \emptyset$, those tokens of $S(\alpha \wedge \neg\alpha)$ are not executable on any possible world.

Negated action types within P

The semantics is defined the same as before. It is easy to check that formulas from **DLSP**$^-$ are still valid.

Now, let us look closer at the behavior of the negated type within **P**.

Proposition

*For any $\alpha \in$ **AT** and any pointed deontic model $\mathcal{M}, w$,*

$$\begin{array}{lcl} \mathcal{M}, w \vDash \mathbf{P}\neg\alpha & \iff & \textit{there is a } v \in W \textit{ s.t. } wRv \textit{ and } A(\alpha, v) = \varnothing \\ \mathcal{M}, w \vDash \mathbf{P}\neg\neg\alpha & \iff & \textit{there is a } v \in W \textit{ s.t. } wRv \textit{ and } A(\alpha, v) \neq \varnothing. \end{array}$$

Intuitively, it is permitted not to do α ($\mathbf{P}\neg\alpha$) iff on *some* deontically ideal world *no* token of α is executed. On the other hand, $\mathbf{P}\neg\neg\alpha$, as a $\Diamond\exists$ (or equivalently $\exists\Diamond$) bundle, is intuitively weaker than **P** as a $\forall\Diamond$ bundle. It naturally provides us with a notion of *weak permission* $\mathbf{P}^w\alpha := \mathbf{P}\neg\neg\alpha$, which corresponds to the permission as $\Diamond$ in $\mathbb{SDL}$.

Some validities

Proposition

For any $\alpha, \beta \in \mathbf{AT}$*, the following schemata are valid:*

$$\mathbf{P}\alpha \rightarrow \mathbf{P}\neg\neg\alpha \qquad \mathbf{P}\neg\neg\neg\alpha \leftrightarrow \mathbf{P}\neg\alpha \qquad \neg\mathbf{P}(\alpha \wedge \neg\alpha).$$

$\mathbf{P}\alpha \rightarrow \mathbf{P}\neg\neg\alpha$ says that strong permission implies weak permission, but not the other way around. From a technical point of view, the behavior of action negation in the scope of **P** is intuitionistic: though double negation only holds in one way, triple negation is equivalent to single negation. For impossible action type $(\alpha \wedge \neg\alpha)$, permission to do it is never issued.

Some invalidities

Proposition

For any $\alpha, \beta \in \mathbf{AT}$, *the following schemata are* ***not*** *valid:*

$$\mathbf{P}\neg\neg\alpha \to \mathbf{P}\alpha \qquad \neg\mathbf{P}\alpha \to \mathbf{P}\neg\alpha \qquad \mathbf{P}\neg\alpha \to \neg\mathbf{P}\alpha \qquad \mathbf{P}(\alpha \vee \neg\alpha).$$

Note that on singleton models, $\mathbf{P}\neg\neg\alpha \to \mathbf{P}\alpha$ is valid when α is a conjunction of literals. Further, the permission to do α is compatible with the permission not to do it. It means that $\mathbf{P}\alpha$ and $\mathbf{P}\neg\alpha$ can be both true thus it is equivalent to $\mathbf{P}(\alpha \vee \neg\alpha)$ by free choice, which is *not* a trivial permission at all: you may do any action of α or simply choose not to do it.

De Morgan's laws in the scope of P

Interestingly, only three of four one-way De Morgan's laws are valid in the scope of **P**, in parallel to the behavior of distribution laws.

Proposition

For any $\alpha, \beta \in \mathbf{AT}$, *we have:*

1. $\models \mathbf{P}\neg(\alpha \vee \beta) \leftrightarrow \mathbf{P}(\neg\alpha \wedge \neg\beta)$;
2. $\models \mathbf{P}\neg(\alpha \wedge \beta) \leftrightarrow (\mathbf{P}\neg\alpha \vee \mathbf{P}\neg\beta)$; *hence, by free choice* $\models \mathbf{P}(\neg\alpha \vee \neg\beta) \rightarrow \mathbf{P}\neg(\alpha \wedge \beta)$ *and* $\not\models \mathbf{P}\neg(\alpha \wedge \beta) \rightarrow \mathbf{P}(\neg\alpha \vee \neg\beta)$.

Invalid $\mathbf{P}\neg(\alpha \wedge \beta) \rightarrow \mathbf{P}(\neg\alpha \vee \neg\beta)$: consider the following case of a course with student presentations and final essays. Suppose the students are permitted *not* to do both. It seems to be consistent with the additional fact that handing in the essay is compulsory, but skipping the presentation is fine. This is a case when $\mathbf{P}\neg(\alpha \wedge \beta) \wedge \neg\mathbf{P}\neg\alpha$ is consistent.

Definability of other deontc modalities using $\mathbf{P}$ and $\neg$

With the action negation $\neg$ at hand, following the usual definitions in $\mathbb{SDL}$, we define $\mathbf{O}\alpha := \neg\mathbf{P}\neg\alpha$ and $\mathbf{F}\alpha := \mathbf{O}\neg\alpha$. Recall that we have defined weak permission $\mathbf{P}^w\alpha := \mathbf{P}\neg\neg\alpha$.

Here, $\mathbf{O}$ defined as $\neg\mathbf{P}\neg$ appears to be the *dual* of $\mathbf{P}$. However, two negations in and out of the scope of $\mathbf{P}$ are different:

- $\neg$ outside $\mathbf{P}$ is for proposition and classical.
- $\neg$ inside $\mathbf{P}$ is for action type and essentially non-classical.

So, the apparent duality is different from that in $\mathbb{SDL}$ and other similar frameworks where no distinction as such is made.

Comparison with $\mathbb{SDL}$

The truth conditions for **O**, **F** and $\mathbf{P}^w$ are as follows:

Proposition

For any $\alpha \in \mathbf{AT}$ *and any pointed deontic model* $\mathcal{M}, w$*,*

$\mathcal{M}, w \vDash \mathbf{O}\alpha$	$\iff$	*for any* $v \in W$*, if* wRv *then* $A(\alpha, v) \neq \varnothing$
$\mathcal{M}, w \vDash \mathbf{F}\alpha$	$\iff$	*for any* $v \in W$*, if* wRv *then* $A(\alpha, v) = \varnothing$.
$\mathcal{M}, w \vDash \mathbf{P}^w\alpha$	$\iff$	*there is a* $v \in W$ *s.t.* wRv *and* $A(\alpha, v) \neq \varnothing$.

We claim that **O** as $\Box\exists$, **F** as $\neg\Diamond\exists$ and $\mathbf{P}^w$ as $\Diamond\exists$ in our framework behaves exactly as the (weak) obligation $\Box$, prohibition $\neg\Diamond$ and weak permission $\Diamond$ in $\mathbb{SDL}$, respectively. So, we can define standard deontic modalites using strong permission **P** and action negation $\neg$.

Proposition

$\vDash \mathbf{O}\alpha \leftrightarrow \neg\mathbf{P}^w\neg\alpha$*,* $\vDash \mathbf{P}^w\alpha \leftrightarrow \neg\mathbf{O}\neg\alpha$*, and* $\vDash \mathbf{P}^w\alpha \leftrightarrow \neg\mathbf{F}\alpha$*.*

Complete system with $\mathbf{O}, \mathbf{F}, \mathbf{P}$ without negated actions

(TAUT)	Propositional Tautologies
(FC)	$\mathbf{P}(\alpha \vee \beta) \leftrightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$
(CE)	$\mathbf{P}(\alpha \wedge \beta) \rightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$
($\mathtt{COM}_\wedge$)	$\mathbf{P}(\alpha \wedge \beta) \leftrightarrow \mathbf{P}(\beta \wedge \alpha)$
($\mathtt{ASSO}_\wedge$)	$\mathbf{P}((\alpha \wedge \beta) \wedge \gamma) \leftrightarrow \mathbf{P}(\alpha \wedge (\beta \wedge \gamma))$
(CD)	$\mathbf{P}(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow \mathbf{P}((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
(CAO)	$(\mathbf{O}\alpha \wedge \mathbf{O}\beta) \rightarrow \mathbf{O}(\alpha \wedge \beta)$
(IFCF)	$(\mathbf{F}\alpha \wedge \mathbf{F}\beta) \rightarrow \mathbf{F}(\alpha \vee \beta)$
(OFO)	$(\mathbf{O}(\alpha \vee \beta) \wedge \mathbf{F}\beta) \rightarrow \mathbf{O}\alpha$
(FOF)	$(\mathbf{F}(\alpha \wedge \beta) \wedge \mathbf{O}\beta) \rightarrow \mathbf{F}\alpha$
(SW)	$\mathbf{P}\alpha \rightarrow \neg\mathbf{F}\alpha$
(SE)	$\mathbf{O}\alpha \rightarrow \neg\mathbf{F}\alpha$
(MP)	Given φ and $(\varphi \rightarrow \psi)$, infer ψ.
(MOO)	Given $\vdash \alpha \rightarrow \beta$, infer $\vdash \mathbf{O}\alpha \rightarrow \mathbf{O}\beta$.
(IMOF)	Given $\vdash \alpha \rightarrow \beta$, infer $\vdash \mathbf{F}\beta \rightarrow \mathbf{F}\alpha$.

Complete system with negated actions over singleton models

On top of the previous system:

	System $\mathbb{POFL}^-$ with the axioms
(PNF)	$\mathbf{P}(\overline{p_i}^n \wedge \overline{\neg\gamma_j}^m) \leftrightarrow \neg\mathbf{F}(\overline{p_i}^n \wedge \overline{\neg\gamma_j}^m)$
(FON)	$\mathbf{F}\alpha \leftrightarrow \mathbf{O}\neg\alpha$
(Fbot)	$\mathbf{F}(\alpha \wedge \neg\alpha)$

To avoid Ross's paradox for obligation, we define $\mathbf{O}^s\alpha := \mathbf{O}\alpha \wedge \mathbf{P}\alpha$.

$\mathbf{P}$	$\mathbf{O}$	$\mathbf{F}$	$\mathbf{O}^s$	$\mathbf{P}^w$
$\forall\Diamond$	$\Box\exists$	$\neg\Diamond\exists$	$\forall\Diamond \wedge \Box\exists$	$\Diamond\exists$

Check our DEON25 paper for details.

Conclusions and future work

Conclusions

- We formalize strong permission as a $\forall x \Diamond$ bundle
- The propositional formulas are action types whose tokens are given by a BHK-style recursive definition
- The resulting logic admits FC and most other good properties, if not all.
- It also predicts phenomena aligned with our linguistic intuition, which were not discussed in the literature
- We can add higher-order permission and other connectives.
- Add action negation $\neg$ as not being done to the framework and define other standard deontic operators using it and $\mathbf{P}$.
- Axiomatize logics with $\mathbf{P}$, $\mathbf{O}$ and $\mathbf{F}$ under various classes of serial deontic models

Ongoing work

- Axiomatizing the logic without the single space assumption.
- Adding implications in the scope of modalities
- Adding sequential "and" such that the tokens are sequences
- Try to solve more puzzles!

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