Bundles in Epistemic Logic

Epistemic Logics of Know-wh

Yanjing Wang
wangyanjing.com
Department of Philosophy, Peking University

NASSLLI25, UW

Logics of know-wh

Beyond knowing that

Knowledge is not only expressed in terms of *"knowing that"*:

- I *know whether* I can get rid of the jet lag this time.
- I *know what* the meaning of the world is.
- I *know how* to miss my own lecture.
- I don't *know why* they started weeding that early.
- I don't *know who* will come to my lecture tomorrow.
- I don't *know how* to sleep well but I *know that* my wife *knows how* and I *know why* she knows.

Hits (in millions) returned by google:

X	that	whether	what	how	who	why
"know X"	574	28	592	490	112	113
"knows X"	50.7	0.51	61.4	86.3	8.48	3.55

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Linguistically: why can't we replace "know" by "believe"?

Philosophically: reduciblility to "knowledge-that"?

Logically: how to reason about "know-wh"?

Computationally: efficient representation and reasoning

What about the logic part?

Epistemic logic is a major subfiled of modal logic initiated by von Wright and Hintikka, which has a wide range of applications in TCS, AI, game theory beyond philosophy.

knowledge-that	—	propositional modal logic
knowledge-wh	—	quantified modal logic

"knowing who" was discussed by Hintikka (1962) in terms of first-order modal logic, e.g., *knowing who* murdered Mary:

$$\exists x \mathsf{K} M(x, Bob).$$

Compare it with $\mathsf{K}\exists x M(x, Bob)$: *de re* vs. *de dicto*.

Knowledge-wh is in general *de re* (knowledge of things).

See my survey paper *Beyond knowing that: a new generation of epistemic logics*, for the early contributions of Hintikka.

The neglected topic of know-wh (until recently)

Quantified modal logic is infamous for its various philosophical and technical "problems", and was under developed.

The early scattered discussions on know-wh seem to be largely forgotten in the later literature, for example:

- In the *Handbook of Epistemic Logic* (2015), there is hardly anything explicitly about quantified epistemic logic nor logic of know-wh (except epistemic strategic logic).
- In the very same paper where public announcement logic was proposed, Plaza (1989) actually spent half of the paper discussing knowing what (the value is).
- The same operator was defined and discussed earlier by Xiwen Ma and Weide Guo from Peking University (IJCAI 83).

We will come back to Plaza's paper later on.

Some developments for FO epistemic logic

A slightly out-dated survey in Gochet and Gribomont (2006)

Mostly application-driven (not an exhaustive list):

- about games: Kaneko and Nagashima (1996)
- about cryptographic knowledge: Cohen and Dam (2007)
- about security protocols: Belardinelli and Lomuscio (2011)
- (un)decidability: Wolter (2000), Sturm et al (2000)
- *de dicto* vs. *de re*: distinction Corsi and Orlandelli (2011)
- "second-order" epistemic logic: Belardinelli and van der Hoek (2015, 2016)
- ...

Beyond knowing that: starting point

Instead of using the full language of quantified modal logic, we can use some well-behaved *fragments* of it to focus on what we really care but no more.

Can we repeat the success of propositional modal logic by a systematic approach to know-wh?

- simple language
- intuitive semantics
- useful models
- balanced expressive power and complexity...

Bundles can help!

The proposal of the "bundled" approach [Wang18]

- take a know-wh construction as a single modality (a "bundle"), e.g., pack $\exists x \mathsf{K}(Mary \approx x)$ into *Kwho Mary*
- the use of quantifiers is restricted (recall the secret of success of propositional modal logic).
- natural and succinct to express the desired properties, e.g., I *know that* you *know what* the password is but I do not know the password.
- capture the essence of the relevant reasoning by axioms.
- stay (technically) neutral for certain philosophical issues.

For each know-wh: the work flow

- focus on some logically interesting types of know-wh;
- find the right bundle as the semantics, guided by philosophical and linguistic theories;
- axiomatize logics with (combinations of) new modalities;
- simplify the semantics while keeping the validities;
- capture the expressivity via notions of bisimulation;
- dynamify those logics with new updates of knowledge;
- automate the inferences based on decidability.
- probably come back to philosophy and linguistics with new insights and questions.

Some earlier know-wh logics we studied

wh-word	bundle (roughly)	connection	key ref
whether	$Kw\varphi := \mathsf{K}\varphi \vee \mathsf{K}\neg\varphi$	non-contingency logic	[FWvD14,15
what	$Kvc := \exists x \mathsf{K}(x \approx c)$	weakly aggregative logic	[WF13,14]
how	$Kh\varphi := \exists \sigma \mathsf{K}[\langle\sigma\rangle]\varphi$	game logic, ATL	[Wang15,17
why	$Ky\varphi := \exists t \mathsf{K}(t : \varphi)$	justification logic	[XWS18]

We obtained complete axiomatizations, characterizations of expressive power, simplified semantics, and decidability ...

See my NASSLLI18 course slides for details `wangyanjing.com/beyond-knowing-that/`. For an updated survey see Section 4 of SEP entry of Epistemic Logic.

Example: A logic of knowing how to achieve [IJCAI17]

TAUT	all axioms of propositional logic	MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
DISTK	$\mathsf{K}p \land \mathsf{K}(p \to q) \to \mathsf{K}q$	NECK	$\dfrac{\varphi}{\mathsf{K}\varphi}$
T	$\mathsf{K}p \to p$	MonoKh	$\dfrac{\varphi \to \psi}{\mathsf{Kh}\varphi \to \mathsf{Kh}\psi}$
4	$\mathsf{K}p \to \mathsf{K}\mathsf{K}p$	SUB	$\dfrac{\varphi(p)}{\varphi[\psi/p]}$
5	$\neg\mathsf{K}p \to \mathsf{K}\neg\mathsf{K}p$		
AxKtoKh	$\mathsf{K}p \to \mathsf{Kh}p$		
AxKhtoKKh	$\mathsf{Kh}p \to \mathsf{K}\mathsf{Kh}p$		
AxKhtoKhK	$\mathsf{K}\mathsf{Kh}p \to \mathsf{Kh}\mathsf{K}p$		
AxKhKh	$\mathsf{Kh}\mathsf{Kh}p \to \mathsf{Kh}p$		
AxKhbot	$\neg\mathsf{Kh}\bot$		

Typical technical difficulties introduced by the bundles

- (apparently) not normal:
  - $\not\models Kw(p \wedge q) \to (Kw\, p \wedge Kw\, q)$
  - $\not\models Kh\varphi \wedge Kh\psi \to Kh(\varphi \wedge \psi)$
  - $\models \varphi \not\Rightarrow\, \models Ky\varphi$
- not strictly weaker either: $\models Kw\varphi \leftrightarrow Kw\neg\varphi$;
- alternation of quantifiers and modalities, e.g., $\exists x \Box \varphi(x)$;
- the things we quantify sometimes have structures;
- the axioms depend on the shape of φ as well;
- weak language vs. rich model: hard to axiomatize;
- fragments of FO/SO-modal language: we know little.

We will give you a list of tips at the end of the lecture.

Connections to existing logics and linguistic theories

Classification by question words:

- Knowing whether: non-contingency logic, ignorance logic
- Knowing what: weakly aggregative logic, dependence logic
- Knowing how: game Logic, alternating temporal logic
- Knowing why: (quantified) justification Logic
- Knowing who: (dynamic) termed modal logic

Classification by logical forms:

- *Mention-some*: e.g., *knowing how/why*... $\exists x \mathsf{K} \varphi(x)$
- *Mention-all* (strongly exhaustive reading): e.g., I *know who* came to the party... $\forall x(\mathsf{K}\varphi(x) \vee \mathsf{K}\neg\varphi(x))$
- *In-between*: *know-value* $\exists x(\mathsf{K}\ c \approx x) \leftrightarrow \forall x(\mathsf{K}\ c \approx x \vee \mathsf{K}\ c \not\approx x)$

Epistemic logic: form one to many

(Routine) research questions:

- Model theory, proof theory, computational complexity
- Group knowledge
- Logical omniscience
- Natural dynamics
- Applications

New questions:

- Interactions of different knowledge expressions;
- Simplification of semantics.
- Epistemology questions...

Knowing value as a minimal example

A classic paper in Dynamic Epistemic Logic (DEL)

- Jan Plaza: Logics of public communications. In Proceedings of the 4th ISMIS Oak Ridge, pp. 201-216. (1989) Unknown for a long time.
- Rediscovered in the late 90s after Gerbrandy and Groeneveld (1997) proposed a similar logic independently (in the Amsterdam tradition of update semantics).
- Reprinted in *Synthese* Volume 158, Issue 2, pp 165-179 (2007), with Hans van Ditmarsch's comments about the history of **DEL** before and after Plaza's paper, and content of the paper (pp 181-187).

"Classic" - a book which people praise and don't read.

– Mark Twain

What Plaza did

- Syntax and semantics of *public announcement logic* (**PAL**):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathsf{K}_i\varphi \mid \mathsf{Kw}_i\varphi \mid \varphi + \varphi$$

  $\mathcal{M}, s \vDash \varphi + \psi \Leftrightarrow \mathcal{M}, s \vDash \varphi$ and $\mathcal{M}|_\varphi, s \vDash \psi$, where $\mathcal{M}|_\varphi$ is a submodel of $\mathcal{M}$ collecting all the worlds satisfying φ in $\mathcal{M}$.
- $\varphi + \psi$ is essentially $\langle\varphi\rangle\psi$ in the modern syntax of **PAL**.
- Discover the reduction to epistemic logic
- Give a complete proof system via reduction axioms: e.g., $\varphi + (\psi_1 \wedge \psi_2) \equiv (\varphi + \psi_1) \wedge (\varphi + \psi_2)$

Plaza's notation may help to see reduction axioms

$\varphi + \psi \not\equiv \psi + \varphi, \varphi + \varphi \not\equiv \varphi$ but...

The following are provable theorems:

$$\top + \varphi \equiv \varphi$$

$$\bot + \varphi \equiv \bot$$

$$\varphi + (\psi + \chi) \equiv (\varphi + \psi) + \chi$$

$$\varphi + \psi \to \varphi$$

$$(\varphi_1 + \cdots + \varphi_i + \cdots + \varphi_n) \to (\varphi_1 + \cdots + \varphi_i)$$

$$(\varphi + \psi_1) \wedge (\varphi + (\psi_1 \to \psi_2)) \to \varphi + \psi_2$$

But that is only *half* of the paper!

One of the two running examples in Plaza's paper

Mr. Sum & Mr. Product

Mr. Puzzle: *I choose two natural numbers greater than 1 such that the sum is less than 100. I will tell the sum of the numbers only to Mr. Sum, and their product only to Mr. Product.*

He tells them.

Mr. Product: *I do not know the numbers.*
Mr. Sum: *I knew you didn't.*
Mr. Product: *But now I know!*
Mr. Sum: *So do I!*
What are the two numbers?

How to express *knowing the numbers*?

Know-value operator by Plaza (also Ma & Guo IJCAI83)

ELKv is defined as (where $c \in C$ is a constant symbol):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \land \varphi) \mid \mathsf{K}_i\varphi \mid Kv_i c$$

Kv_i says "agent i knows [what] the value of c [is]"

ELKv is interpreted on FO-epistemic (S5) models with a *constant* domain $\mathcal{M} = \langle S, D, \{\sim_i \mid i \in \mathbf{I}\}, V, V_C\rangle$, where V_C assigns to each (non-rigid) $c \in C$ an $o \in D$ on each $s \in S$:

$$\mathcal{M}, s \vDash Kv_i c \iff \text{for any } t_1, t_2: \text{ if } s \sim_i t_1, s \sim_i t_2, \text{ then } V_C(c, t_1) = V_C(c, t_2).$$

Essentially the semantics is the bundle $\exists x \mathsf{K}(c \approx x)$.

Know-value operator by Plaza (also Ma, Guo IJCAI83)

ELKv can express "*i* knows that *j* knows the password but *i* doesn't know what exactly it is" by $\mathsf{K}_i Kv_j c \wedge \neg Kv_i c$.

The interaction between the two operators is crucial: it cannot be treated as $\mathsf{K}_i \mathsf{K}_j p \wedge \neg \mathsf{K}_i p$ which is inconsistent.

It is crucial in security protocol verification. Ways to capture "knowing what": e.g., introducing $has_i(m)$ as a basic proposition with a database of messages in the semantics.

See [Dechesne & Wang, Synthese 2010] for a survey on various knowledge in the security setting.

Know-value operator by Plaza (also Ma, Guo IJCAI83)

To handle the *Sum and Product* puzzle, Plaza extended **ELKv** with announcement operator (call it **PALKv**):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathsf{K}_i\varphi \mid Kv_i c \mid \langle\varphi\rangle\varphi$$

Plaza mentioned some axioms on top of $\mathbb{S}5$ and van Ditmarsch (2007) raised their completeness as a question.

`Kv4`	$Kv_i c$	$\to$	$\mathsf{K}_i Kv_i c$
`Kv5`	$\neg Kv_i c$	$\to$	$\mathsf{K}_i \neg Kv_i c$
`KKv`	$\langle \mathsf{K}_i\varphi\rangle Kv_i c$	$\leftrightarrow$	$\mathsf{K}_i\varphi \wedge Kv_i c$
	$\langle Kv_i c\rangle Kv_i d$	$\leftrightarrow$	$Kv_i c \wedge Kv_i d$
	$\langle\varphi\rangle Kv_i c$	$\to$	$\mathsf{K}_i(\varphi \to \langle\varphi\rangle Kv_i c)$
	$\langle\varphi\rangle \neg Kv_i c$	$\to$	$\mathsf{K}_i(\varphi \to \langle\varphi\rangle \neg Kv_i c)$

Call $\mathbb{S}5$ plus Plaza's three axioms $\mathbb{PALKV}_p$.

Theorem (Wang & Fan IJCAI13)

$\theta = \langle p \rangle Kv_i c \wedge \langle q \rangle Kv_i c \to \langle p \vee q \rangle Kv_i c$ is not provable in $\mathbb{PALKV}_p$, thus $\mathbb{PALKV}_p$ is not complete.

Proof idea:

- define a class $\mathbb{C}$ of two-dimensional models (with $\xrightarrow{\varphi}$-labelled transitions) and a new semantics $\Vdash$ for **PALKv** such that:
- for all **PALKv** formulas φ: $\vdash \varphi \implies \mathbb{C} \Vdash \varphi$
- show that $\mathbb{C} \nVdash \theta$.

Cf. [Wang & Cao Synthese 2013] for the general method of constructing such semantics for **PAL** and incompleteness.

A bisimulation notion

We can use a notion of bisimulaiton to understand the expressivity of **ELKv**.

A *d-bisimulation* between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a non-empty relation $Z \subseteq S_1 \times S_2$ such that if $s_1 Z s_2$ then the following requirements hold for all $i \in \mathbf{I}$ (besides the standard bis conditions):

Kv-Zig: if $t_1 \sim_i^1 s_1 \sim_i^1 t_1'$, and $V_C^1(c, t_1) \neq V_C^1(c, t_1')$ for some c then there exist $t_2, t_2' \in S_2$ such that $t_2 \sim_i^2 s_2 \sim_i^2 t_2'$, and $V_C^2(c, t_2) \neq V_C^2(c, t_2')$;

Kv-Zag: symmetric

We write $\mathcal{M}_1, s_1 \leftrightarroweq_d \mathcal{M}_2, s_2$ iff there is a d-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ linking s_1 and s_2.

Proposition

If $\mathcal{M}_1, s_1 \leftrightarroweq_d \mathcal{M}_2, s_2$, then $\mathcal{M}_1, s_1 \equiv_{\mathbf{ELKv}} \mathcal{M}_2, s_2$.

A reduction-based axiomatization is impossible

Now consider the following two epistemic models (using $\circ$ and $\bullet$ for the objects assigned to c):

$$s : p \circ \overset{1}{—} \neg p \circ \overset{1}{—} p \bullet \qquad s' : p \circ \overset{1}{—} \neg p \bullet$$

It is not hard to see that these two models are d-bisimiliar linking s and s'. However, we can distinguish s and s' easily by a **PALKv** formula $[p]Kv_1 c$.

Theorem (Wang & Fan IJCAI13)

PALKv *is strictly more expressive than* ***ELKv****.*

Conditionally knowing value

Conditionally knowing what

Axiomatizing **PALKv** looks hard at the beginning. We propose a generalization of Kv_i operator inspired by the relativized common knowledge operator (call it $\mathbf{ELKv}^r$):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{K}_i\varphi \mid Kv_i(\varphi, c)$$

where $Kv_i(\varphi, c)$ says "agent i knows what c is *given* φ", e.g., I know my password for this website given it is 4-digit.

$\mathcal{M}, s \vDash Kv_i(\varphi, c)$	$\Leftrightarrow$	for any $t_1, t_2 \in S$ such that $s \sim_i t_1$ and $s \sim_i t_2$: $\mathcal{M}, t_1 \vDash \varphi \& \mathcal{M}, t_2 \vDash \varphi$ implies $V_C(c, t_1) = V_C(c, t_2)$

The bundle $Kv_i(\varphi, c)$ is $\exists x \mathsf{K}_i(\varphi \to c \approx x)$, thus $\vDash Kv_i c \leftrightarrow Kv_i(\top, c)$.

Let $\mathbf{PALKv}^r$ be:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathsf{K}_i\varphi \mid Kv_i(\varphi, c) \mid \langle\varphi\rangle\varphi$$

PALKv^r looks more expressive than **PALKv** but in fact they are equally expressive.

Theorem (Wang & Fan IJCAI13)

The comparison of the expressive power of those logics are summarized in the following (transitive) diagram:

$$\begin{array}{ccc} \textbf{\textit{ELKv}}^r & \longleftrightarrow & \textbf{\textit{PALKv}}^r \\ \uparrow & & \updownarrow \\ \textbf{\textit{ELKv}} & \longrightarrow & \textbf{\textit{PALKv}} \end{array}$$

Translation t : **ELKv**$^r \to$ **PALKv**, g : **PALKv**$^r \to$ **ELKv**^{r}

$$\begin{array}{rcl} t(Kv_i(\varphi, d)) & = & \mathsf{K}_i \neg t(\varphi) \vee \hat{K}_i \langle t(\varphi) \rangle Kv_i d \\ g(\langle \varphi \rangle Kv_i(\psi, d)) & = & g(\varphi) \wedge g(Kv_i(\langle \varphi \rangle \psi, d)) \end{array}$$

A sound and complete axiomatization

System $\mathbb{ELKV}^r$

Axiom Schemas	
TAUT	all the instances of tautologies
DISTK	$K_i(p \to q) \to (K_i p \to K_i q)$
T	$K_i p \to p$
4	$K_i p \to K_i K_i p$
5	$\neg K_i p \to K_i \neg K_i p$
$\mathtt{DISTKv}^r$	$K_i(p \to q) \to (Kv_i(q, c) \to Kv_i(p, c))$
$\mathtt{Kv}^r 4$	$Kv_i(p, c) \to K_i Kv_i(p, c)$
$\mathtt{Kv}^r \bot$	$Kv_i(\bot, c)$
$\mathtt{Kv}^r \vee$	$\hat{K}_i(p \wedge q) \wedge Kv_i(p, c) \wedge Kv_i(q, c) \to Kv_i(p \vee q, c)$

Rules	
MP	$\dfrac{p, p \to q}{q}$
NECK	$\dfrac{\varphi}{K_i \varphi}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi}$

$\mathtt{Kv}^r \vee$ was inspired by $\theta = \langle p \rangle Kv_i c \wedge \langle q \rangle Kv_i c \to \langle p \vee q \rangle Kv_i c$.

$K_i Kv_i(\varphi, d) \leftrightarrow Kv_i(\varphi, d)$	T, $\mathtt{Kv}^r 4$
$\neg K_i Kv_i(\varphi, d) \to K_i \neg K_i Kv_i(\varphi, d)$	5
$\neg Kv_i(\varphi, d) \to K_i \neg Kv_i(\varphi, d)$	RE

Core ideas for the completeness

$Kv_i(\varphi, c)$ can be viewed as $\exists x \mathsf{K}_i(\varphi \to c \approx x)$ where x is a variable. Weak language vs. rich model.

Bundle means trouble: to build a canonical model, just using maximal consistent sets as building blocks won't work: (Two worlds below satisfy exactly the same formulas but you do need two worlds to satisfy $\neg Kvd$, a single MCS won't work)

$$p, d \mapsto \circ \text{———} p, d \mapsto \bullet$$

We can saturate each maximal consistent set with:

- counterparts of atomic formulas such as $c \approx x$
- counterparts of $\mathsf{K}_i(\varphi \to c \approx x)$

In short, we use some fake formulas (semantic objects).We need to make sure the extra information is "consistent" by some conditions on the MCSs.

Definition (Wang & Fan AiML14)

Let MCS be the set of maximal consistent sets w.r.t. $\mathbb{ELKV}^r$, and let $\mathbb{N}$ be the set of natural numbers. The canonical model $\mathcal{M}$ of $\mathbb{ELKV}^r$ is a tuple $\langle S, \mathbb{N}, \{\sim_i \mid i \in \mathbf{I}\}, V, V_C\rangle$ where:

- S consists of all the triples $\langle \Gamma, f, g\rangle \in MCS \times \mathbb{N}^C \times (\mathbb{N} \cup \{\star\})^{\mathbf{I} \times \mathbf{ELKv}^r \times C}$ that satisfy the following three conditions:

  (i) $g(i, \psi, d) = \star$ iff $Kv_i(\psi, d) \wedge \hat{K}_i\psi \notin \Gamma$,

  (ii) If $g(i, \varphi, d) \neq \star$ and $g(i, \psi, d) \neq \star$ then: $g(i, \varphi, d) = g(i, \psi, d)$ iff $Kv_i(\varphi \vee \psi, d) \in \Gamma$

  (iii) $\psi \wedge Kv_i(\psi, d) \in \Gamma$ implies $f(d) = g(i, \psi, d)$.

- $s \sim_i t$ iff $\{\varphi \mid \mathsf{K}_i\varphi \in s\} \subseteq t$ and $g(i) = g(i)$ in s
- $V_C(d, s) = f(d)$ in s.

f and g are counterparts of $d \approx x, \mathsf{K}_i(\varphi \to d \approx x)$ formulas.

The conditions (i)-(iii) make sure the fake formulas are "consistent" with Γ. To find the right conditions is not easy.

Lemma (Lindenbaum plus)

Each maximal consistent set can be properly saturated with those counterparts.

Lemma

Each saturated MCS including $\hat{K}\varphi$ has a saturated φ-successor.

Lemma (Existence lemma doubled)

Each saturated MCS including $\neg Kv_i(\varphi, c)$ has ***two*** *saturated φ-successors which disagree about the value of c.*

In general, it is much harder for a $\exists x$K bundle: you may need to construct infinitely many successors in know-how/why...

The existence lemma is broken down to two propositions:

Proposition

Given any $s \in S^c$ *and any* $i \in \boldsymbol{I}$, *suppose there exist two (possibly identical) maximal consistent sets* Γ_1 *and* Γ_2 *such that:*

- ***(a)*** $\{\psi \mid \mathsf{K}_i\psi \in s\} \subseteq \Gamma_1 \cap \Gamma_2$
- ***(b)*** *for any* $Kv_i(\theta, d) \in s$, $\theta \notin \Gamma_1 \cap \Gamma_2$.

then Γ_1 *and* Γ_2 *can be extended into two states* w, v *in* S^c *such that* $s \sim_i^c w$, $s \sim_i^c v$ *and* $f_w(d) \neq f_v(d)$.

Proposition

Given any $s \in S^c$ and any $i \in \boldsymbol{I}$, suppose $\neg Kv_i(\varphi, d) \in s$ then there are two (possibly identical) maximal consistent sets Γ_1 and Γ_2 such that:

(a') $\{\varphi\} \cup \{\psi \mid \mathsf{K}_i\psi \in s\} \subseteq \Gamma_1 \cap \Gamma_2$

(b) *for any $Kv_i(\theta, d) \in s$, $\theta \notin \Gamma_1 \cap \Gamma_2$.*

Let $Z = \{\psi \mid \mathsf{K}_i\psi \in s\} \cup \{\varphi\}$ and let $X = \{\neg\theta \mid Kv_i(\theta, d) \in s\}$. Note that due to $\mathtt{Kv}^r\bot$, X is non-empty. We want to build two consistent sets B and C such that $Z \subseteq B \cap C$ and $X \subseteq B \cup C$. Let $B_0 = Z \cup \{\neg\theta_0\}$ and let $C_0 = Z$ as the starting points. Then we build B_{n+1} and C_{n+1} based on the already defined B_n and C_n by adding $\neg\theta_{n+1}$ into one of them.

A generalization of Axiom $\mathtt{Kv}^r\vee$ (U is a finite set of formulas) $\hat{K}_i(\bigwedge U) \wedge \bigwedge_{\varphi \in U} Kv_i(\varphi, d) \to Kv_i(\bigvee U, d)$ is crucial.

Completeness proof requires 10+ pages

Theorem (Wang & Fan AiML14)

$\mathbb{ELKV}^r$ *is sound and strongly complete for* ***ELKv***^{r}.

We can axiomatize multi-agent **PALKv**^{r} by adding the following reduction axiom schemas (call the resulting system $\mathbb{SPALKV}^r$):

!ATOM	$\langle\psi\rangle p \leftrightarrow (\psi \wedge p)$
!NEG	$\langle\psi\rangle\neg\varphi \leftrightarrow (\psi \wedge \neg\langle\psi\rangle\varphi)$
!CON	$\langle\psi\rangle(\varphi \wedge \chi) \leftrightarrow (\langle\psi\rangle\varphi \wedge \langle\psi\rangle\chi)$
!K	$\langle\psi\rangle \mathsf{K}_i\varphi \leftrightarrow (\psi \wedge \mathsf{K}_i(\psi \to \langle\psi\rangle\varphi))$
!Kv^r	$\langle\varphi\rangle Kv_i(\psi, c) \leftrightarrow (\varphi \wedge Kv_i(\langle\varphi\rangle\psi, c))$

If you can prove first-order modal logic is a conservative extension of our logic, you can also obtain completeness (but it is hard as well)

Axiomatizing ELKV^r over arbitrary frames [Ding 2015]

System $\mathbb{ELKV}^r$

Axiom Schemas	
TAUT	all the instances of tautologies
DISTK	$\mathrm{K}_i(p \to q) \to (\mathrm{K}_i p \to \mathrm{K}_i q)$
DISTKv^r	$\mathrm{K}_i(p \to q) \to (Kv_i(q, c) \to Kv_i(p, c))$
Kv$^r\bot$	$Kv_i(\bot, c)$
Kv$^r\vee$	$\hat{K}_i(p \wedge q) \wedge Kv_i(p, c) \wedge Kv_i(q, c) \to Kv_i(p \vee q, c)$

- The SAT problem of this logic is PSPACE-complete over arbitrary models (Ding 2015).

The completeness proofs are highly non-trivial due to the imbalance between the rich model and limited language.

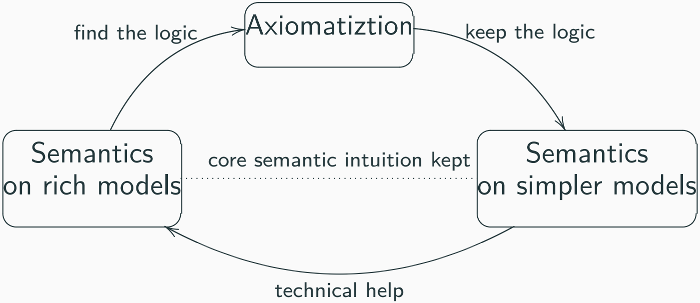
Suitable bisimulation notion for this logic was unknown.

We can do better.

Rematching model with the language

Two questions and our key observation

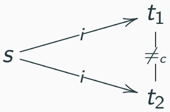
- How to rebalance the syntax and semantics?
- How can it be connected to (normal) modal logic?



Simplify the semantics while keeping the logic [Gu & Wang 16]

Observation: $\neg Kv_i(\varphi, c)$ can be viewed as a special diamond:

$$\mathcal{M}, s \vDash \neg Kv_i(\varphi, c) \quad \Leftrightarrow \quad \text{there exist } t_1, t_2 \in S \text{ such that } s \sim_i t_1 \text{ and } s \sim_i t_2 : \\ \mathcal{M}, t_1 \vDash \varphi \text{ and } \mathcal{M}, t_2 \vDash \varphi \text{ but } V_C(c, t_1) \neq V_C(c, t_2)$$



We do not care about the exact values of c!

Then why not make it a ternary relation?

A modal language

To facilitate the comparison, we write $\neg Kv_i(\varphi, c)$ as $\Diamond_i^c \varphi$ and use the following language $\mathbf{MLKv}^r$:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_i \varphi \mid \Diamond_i^c \varphi$$

interpreted on Kripke models with binary and **ternary** relations $\langle S, \{\to_i : i \in I\}, \{R_i^c : i \in \mathbf{I}, c \in \mathbf{D}\}, V \rangle$, with extra conditions.

$M, s \Vdash \Diamond_i^c \varphi \iff \exists\, u, v$: s.t. $sR_i^c uv$ and $\mathcal{M}, u \Vdash \varphi$, $\mathcal{M}, v \Vdash \varphi$.

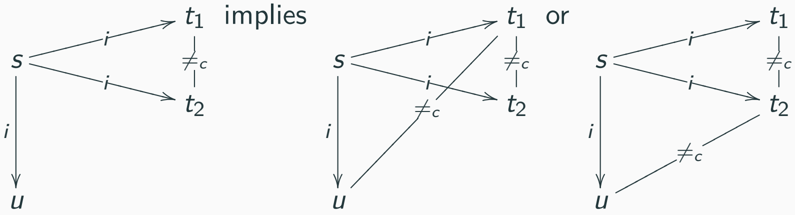
(1) $sR_i^c tu \iff sR_i^c ut$; (2) $sR_i^c uv$ only if $s \to_i u$ and $s \to_i v$;

(3) $sR_i^c tu$ and $s \to_i v$ implies that $sR_i^c tv$ or $sR_i^c uv$ holds;

(4) $sR_j^c tu$ for some $j \in \mathbf{I}$, $s \to_i t$ and $s \to_i u$ implies $sR_i^c tu$;

(5) $sR_j^c tu$ implies $t \neq u$.

$sR_i^c t_1 t_2$ and $s \to_i u$ implies that at least one of $sR_i^c t_1 u$ and $sR_i^c t_2 u$ holds



We show that (4)(5) do not matter: For any set $\Gamma \cup \{\varphi\}$ of **MLKv**^{r} formulas: $\Gamma \Vdash_{\mathbb{C}_{1-5}} \varphi \iff \Gamma \Vdash_{\mathbb{C}_{1-3}} \varphi \iff t(\Gamma) \vDash t(\varphi)$ where t translates **MLKv**^{r} formulas back to **ELKv**^{r}.

Recall the system for ELKV^r.

System $\mathbb{ELKV}^r$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\mathsf{K}_i(p \to q) \to (\mathsf{K}_i p \to \mathsf{K}_i q)$
DISTKv^r	$\mathsf{K}_i(p \to q) \to (Kv_i(q, c) \to Kv_i(p, c))$
Kv$^r\bot$	$Kv_i(\bot, c)$
Kv$^r\vee$	$\hat{K}_i(p \wedge q) \wedge Kv_i(p, c) \wedge Kv_i(q, c) \to Kv_i(p \vee q, c)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\mathsf{K}_i\varphi}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi}$

In the new language:

- DISTKv^r: $\Box_i(p \to q) \to (\Box_i^c \neg q \to \Box_i^c \neg p)$ equivalent to $\Box_i(p \to q) \to (\Box_i^c p \to \Box_i^c q)$ under SUB and RE.
- Kv$^r\vee$: $\Diamond_i(p \wedge q) \wedge \Diamond_i^c(p \vee q) \to (\Diamond_i^c p \vee \Diamond_i^c q)$
- Kv$^r\bot$: $\Box_i^c \top$

A new look at the axiomatization

System $\mathbb{SMLKVR}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
DISTKv^r	$\Box_i(p \to q) \to (\Box_i^c p \to \Box_i^c q)$
Kv$^r\vee$	$\Diamond_i(p \wedge q) \wedge \Diamond_i^c(p \vee q) \to (\Diamond_i^c p \vee \Diamond_i^c q)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i \varphi}$
NECK^r	$\dfrac{\varphi}{\Box_i^c \varphi}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$

We replace $\Box_i^c \top$ by a necessitation rule NECK^r.

Theorem (Gu & Wang AiML16)

$\mathbb{SMLKVR}$ *is sound and complete w.r.t.* $\mathbb{C}_{1-3}$ *(and* $\mathbb{C}_{1-5}$*).*

A relatively easy canonical model construction suffices (3 pages).

A new look at the axiomatization

System $\mathbb{SMLKVR}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
DISTKv^r	$\Box_i(p \to q) \to (\Box^c_i p \to \Box^c_i q)$
Kv$^r\vee$	$\Diamond_i(p \wedge q) \wedge \Diamond^c_i(p \vee q) \to (\Diamond^c_i p \vee \Diamond^c_i q)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i \varphi}$
NECK^r	$\dfrac{\varphi}{\Box^c_i \varphi}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$

Note that $\Diamond^c_i(\varphi \vee \psi) \to (\Diamond^c_i \varphi \vee \Diamond^c_i \psi)$ does not hold.

Moreover, $\Box^c_i(\varphi \to \psi) \to (\Box^c_i \varphi \to \Box^c_i \psi)$ does not hold either, thus the logic is **not** a normal modal logic.

However, this is only the appearance.

Disguised normal modal logic, qua expressive power

$\Diamond_i^c$ is essentially a **binary** diamond!

In MLKvr we only allow $\Diamond_i^c(\varphi, \varphi)$. Let MLKvb be the language with $\Diamond_i^c(\varphi, \psi)$.

$\Diamond_i^c(\varphi, \psi)$ has the standard semantics for (polyadic) normal modal logic:

$$M, s \Vdash \Diamond_i^c(\varphi, \psi) \iff \exists\, u, v\text{: s.t. } sR_i^c uv \text{ and } \mathcal{M}, u \Vdash \varphi,\ \mathcal{M}, v \Vdash \psi.$$

The generalization does not increase expressivity

Proposition

MLKvb *is equally expressive as* MLKvr *over* $\mathbb{C}_{1-3}$.

$\Diamond_i^c(\varphi, \psi)$ is equivalent to the disjunction of the following:

- $\Diamond_i^c\varphi \wedge \Diamond_i\psi$
- $\Diamond_i^c\psi \wedge \Diamond_i\varphi$
- $\Diamond_i\varphi \wedge \Diamond_i\psi \wedge \neg\Diamond_i^c\varphi \wedge \neg\Diamond_i^c\psi \wedge \Diamond_i^c(\varphi \vee \psi)$

A normal polyadic modal logic

System $\mathbb{SMLKVB}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
DISTBK	$\Box_i^c(p \to q, r) \to (\Box_i^c(p, r) \to \Box_i^c(q, r))$
SYM	$\Box_i^c(p, q) \to \Box_i^c(q, p)$
INCL	$\Diamond_i^c(p, q) \to \Diamond_i p$
DISBK	$\Diamond_i^c(p, q) \wedge \Diamond_i r \to \Diamond_i^c(p, r) \vee \Diamond_i^c(q, r)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i \varphi}$
NECKvb	$\dfrac{\varphi}{\Box_i^c(\varphi, \psi)}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$

Theorem (Gu & Wang AiML16)

$\mathbb{SMLKVB}$ *is sound and complete w.r.t.* $\mathbb{C}_{1-3}$ *and* $\mathbb{C}_{1-5}$.

$\mathbb{SMLKVB}$ can drive all the axioms in $\mathbb{SMLKVR}$.

The completeness proof is now mostly routine (one page)

$\mathcal{M}^c = \langle S, \{\rightarrow_i : i \in \mathbf{I}\}, \{R_i^c : i \in \mathbf{I}, c \in \mathbb{C}\}, V\rangle$

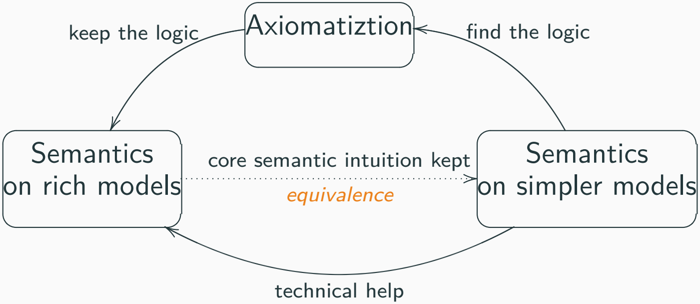
- S is the set of all maximal $\mathbb{SMLKVB}$-consistent sets of `MLKvb` formulas,
- $s \rightarrow_i t \iff \{\varphi : \Box_i \varphi \in s\} \subseteq t$,
- $sR_i^c tu \iff$ (1) $\{\varphi : \Box_i \varphi \in s\} \subseteq t \cap u$ and (2) for any $\Box_i^c(\varphi, \psi) \in s$, $\varphi \in t$ or $\psi \in u$.
- $V(s) = \{p : p \in s\}$.

`SYM`, `INCL`, and `DISBK` are canonical for the corresponding properties 1-3.

If you can find the right abstract model...

Completeness proof: 10 pages ⇒ 3 pages ⇒ (equ. exp.) 1 page

Of course, the burden is then to show the equivalence between the semantics over original model and the abstract model. You can then reserve the work flow:



The original semantics + model is always the starting point.

ELKvr as a normal modal logic

$\mathbf{ELKv}^r$ can be viewed as a disguised normal modal logic!

Standard techniques apply:

- Canonical model for free.
- Bisimulation for free.
- Polyadic modal logic is under-developed too...

(If you like modal logic, do FOML or polyadic modal logic!)

These will help us in solving problems about the original $\mathbf{ELKv}^r$.

Definition (Polyadic Bisimulation)

Let $\mathcal{M}_1 = \langle S_1, \{\rightarrow_i^1 : i \in I\}, \{R_i^c : i \in I, c \in \mathbb{C}\}, V_1\rangle$, $\mathcal{M}_2 = \langle S_2, \{\rightarrow_i^2 : i \in I, c \in \mathbb{C}\}, \{Q_i^c : i \in I\}, V_2\rangle$ be two models for **MLKvb** (also for **MLKv**^{r}). A $\mathbb{C}$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a non-empty binary relation $Z \subseteq S_1 \times S_2$ such that for all $s_1 Z s_2$, the following conditions are satisfied:

Inv : $V_1(s_1) = V_2(s_2)$;

Zig : $s_1 \rightarrow_i^1 t_1 \Rightarrow \exists t_2$ such that $s_2 \rightarrow_i^2 t_2$ and $t_1 Z t_2$;

Zag : $s_2 \rightarrow_i^2 t_2 \Rightarrow \exists t_1$ such that $s_1 \rightarrow_i^1 t_1$ and $t_1 Z t_2$;

Kvb-Zig : $s_1 R_i^c t_1 u_1 \Rightarrow \exists t_2, u_2 \in S_2$ such that $t_1 Z t_2$, $u_1 Z u_2$ and $s_2 Q_i^c t_2 u_2$;

Kvb-Zag : $s_2 Q_i^c t_2 u_2 \Rightarrow \exists t_1, u_1 \in S_1$ such that $t_1 Z t_2$, $u_1 Z u_2$ and $s_1 R_i^c t_1 u_1$.

Can be translated back to **ELKv**^{r}.

A simpler logic

Plaza's unconditional language:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathsf{K}_i\varphi \mid Kv_i c$$

is essentially ($\neg Kv_i c$ is $\Diamond_i^c \top$):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_i\varphi \mid \Box_i^c \bot$$

System $\mathbb{SMLKV}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
INCLT	$\Diamond_i^c \top \to \Diamond_i \top$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i\varphi}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$

For logics over epistemic (S5) models, you need the usual axiom of S5 and $\Diamond_i \Diamond_i^c \top \to \Diamond_i^c \top$ (i.e., $Kv_i c \to \mathsf{K}_i Kv_i c$).

Further directions

Axiomatization of PALKv directly

See Bo Hong's master thesis (2022).

It can be obtained by translating the **PALKv**^{r} axioms and provide some supplementations. The proof can be done by taking $[\varphi]Kvd$ as an atomic formula like $Kv(\varphi, d)$.

We can also extend *Kvc* to *KvP* (knowing a predicate) with mention-some and mention-all semantics [Hong LORI23].

Adding *de re* dynamics

knowing *that*	announcing *that*
knowing *what*	announcing *what*

Enrich **ELKv**^{r} with *public inspection* $[c]$:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathsf{K}_i\varphi \mid Kv_i(\varphi, c) \mid [c]\varphi$$

$$\boxed{\mathcal{M}, s \vDash [c]\varphi \quad \Leftrightarrow \quad \mathcal{M}|^s_c, s \vDash \varphi}$$

where $\mathcal{M}|^s_c$ is defined as the tuple $\langle S', D, \sim|_{S' \times S'}, V|_{\mathbf{D} \times S'}, V_{\mathbf{D}}|_{S'}\rangle$
where $S' = \{s' \mid V_{\mathbf{D}}(c, s') = V_{\mathbf{D}}(c, s)\}$.

Note that the relativization here is *local*.

PSELKv^r is more expressive than **ELKv**^{r}: $[c]$ cannot be reduced.

How to axiomatize it?

It is still open, but you can:

- Restrict the language: to only allow Kv_ic and $[c]\varphi$ [van Eijck, Gattinger, Wang ICLA17]
- Enrich the language: with equalities, and conditional operator for all the combinations of values and propositions [Baltag AiML16]

There is another (much better) way: breaking the bundles into different pieces [Cohen, Tang, Wang TARK21]!

Weakly Aggregative Logic

Modal logic with diagnol modalities as special cases on polyadic modal logic:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

It can be defined on models with n-ary relations.

$\mathcal{M}, w \models \Box\varphi$	iff	for all $v_1, \ldots v_n \in W$ with $Rwv_1 \ldots, v_n, \mathcal{M}, v_i \models \varphi$ for some $i \leq n$.
$\mathcal{M}, w \models \Diamond\varphi$	iff	there are $v_1, \ldots v_n \in W$ st. $Rwv_1 \ldots, v_n$ and $\mathcal{M}, v_i \models \varphi$ for all $i \leq n$.

The following is valid over models with $n+1$-ary relations:

$$\Box p_0 \wedge \cdots \wedge \Box p_n \to \Box \bigvee_{(0 \leq i < j \leq n)} (p_i \wedge p_j).$$

It does not have Craig interpolation! [Liu, Ding and Wang 2019]

WAML over hypergraphs [Ding, Liu, Wang 21]

We can actually define the semantics over hypergraphs, which are a collection of subsets (hyperedges) of a given non-empty set of vertices. We can then define:

$$\mathcal{M}, w \models \Box\varphi \quad \text{iff} \quad \text{for any hyperedges } E \text{ such that } w \in E, \; \mathcal{M}, v \models \varphi \text{ for some } v \in E$$

Note the similarity with neighborhood semantics.

It has close connections with evidence logic, local reasoning, logic of somebody knows.

Temporal logic with Kv [Lin LORI21, thesis 22]

We can combine ETL with the the know-value operator:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid X\varphi \mid \varphi U\psi \mid K_i\varphi \mid Kv_i d \mid C\varphi$$

The semantics is based on epistemic temporal models with a constant domain and value assignments.

We can also introduce the commonly know-value operator Cv:

$$\mathcal{M}, w \vDash Cvd \iff \exists x \in D, \mathcal{M}, w \vDash C(x \approx d).$$

Over S5 models with finite agents Cvd is equivalent to $C \bigwedge_i Kv_i d$

Lin (2022) gave a complete proof system.

Conclusions

Handling the bundling: Problems and Possible Solutions

Mainly about axiomitization.

- P Don't have the desired subformulas in the language
- S Fake it until you make it...
- P Maximal consistent sets are not enough
- S Adding additional information (like fake formulas), find some notion of "consistency" between it and the MCS. Do not forget to prove a stronger Lindenbaum lemma
- P Quantifier alternation leads to complication about the existence lemma
- S Find a general method to construct many possible worlds, some experience with polyadic modal logic can help, or prove it directly (not contrapositive)

Handling the bundling: Problems and Solutions

- P Some components of the model are not even mentioned in the language at all
- S Educated guess, often the naive idea roughly works.
- P Rich model vs. simple language
- S Rebalancing with alternative semantics on abstract models, while keeping the logic. Many things would become more familiar and some results are for free.

Difficulties did not show up in this simple case:

- $\exists x$ quantifies over complicated structures
- The bundle is not expressible in a well-known logic.

This is the crucial thing in know-how and know-why.

Be brave and use your imagination!

Read the classics!