



Bundles in Deontic Logic

Bundle+BHK for non-normal modal logic

Yanjing Wang

Department of Philosophy, Peking University

Dec. 18th, 2024

Background

Language and Semantics

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Background

Phenomena in natural language as “Icebergs”

What shall we do when seeing an iceberg?

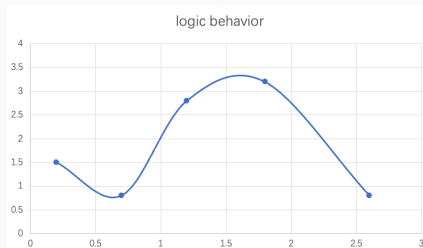


Phenomena in natural language as “Icebergs”

What shall we do when seeing an iceberg?



Using various logic techniques to fit the “data”...

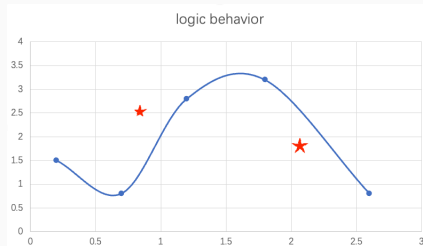


Phenomena in natural language as “Icebergs”

New phenomena by looking at the iceberg from other angles ...



Using logic techniques to fit the **new** “data”...



Data Fitting vs. Understanding Why



Deontic Logic: ocean with lots of icebergs...

There are many logical puzzles in Standard Deontic Logic (SDL), deviating from normal modal logic when taking Obligation (**O**) as a \square and Permission (**P**) as a \diamond .

Among many others:

- Ross' paradox: $\mathbf{O}p \rightarrow \mathbf{O}(p \vee q)$ and $\mathbf{P}p \rightarrow \mathbf{P}(p \vee q)$ are *intuitively invalid*, but *valid* in SDL.
- Free choice: $\mathbf{P}(p \vee q) \rightarrow \mathbf{P}p \wedge \mathbf{P}q$ is *intuitively valid*, but logically *invalid* in SDL.

We focus on *Strong* Permission (**P**), the permissions *explicitly granted* rather than simply not being forbidden. Strong permissions exhibit the property of free choice (FCP).

Basic observation and questions

Deontic modalities may be more than what they appear to be!

- Are they also *bundles* of a quantifier and a usual modality?
- but, *quantifying over what?*

Formulas inside deontic modalities might not be propositions

- Then what are they?
- How can we treat them formally?

Further observations regarding quantifiers and bundles

If a hidden quantifier were present, what would it quantify?

- The distinction between action **types** and **tokens**, i.e., individual actions (well-known in the literature)
- Deontic sentences mention only action **types**
- But the semantics may be about **tokens** of those types

What could be the bundle for strong permission **P**?

- It is clearly **not** $\exists x\Box$, but it might be $\forall x\Diamond$ (Hintikka 1971).
- $P\alpha$: **each** token of action type α is executable on **some** deontically good world of the current world.
- If you are permitted to **take one day off** next week. Each relevant token (taking Monday off, taking Tuesday off...) should be executable on some world.

Further observations: formalizing action type and token

- Propositional **formulas** as action **types**
- They do not have truth values, though can be assigned!
- They can be viewed as collections of action tokens
- We borrow the **BHK-like** formalism to capture them

Brouwer-Heyting-Kolmogorov (BHK) interpretation

BHK *proof* interpretation of *connectives*:

- (H1) A proof of $\alpha \wedge \beta$ is given by presenting a proof of α and a proof of β
- (H2) A proof of $\alpha \vee \beta$ is given by presenting either a proof of α or a proof of β
- (H3) A proof of $\alpha \rightarrow \beta$ is a construction which *transforms* any proof of α into some proof of β
- (H4) Absurdity \perp has *no* proof.

$\neg\alpha$ is the abbreviation of $\alpha \rightarrow \perp$.

We can define the relation between action *tokens and types* recursively like the above.

Further observations: formalizing action type and token

- Propositional **formulas** as action **types**
- They do not have truth values, though can be assigned!
- They can be viewed as a collections of action tokens
- We borrow the BHK-like formalism to capture them

	Intuitionistic Logic	Deontic Logic
prop. formulas	type of problems	type of actions
token	solution/proof	individual act
modality	know-how	permission
bundle	$\exists \Box$	$\forall \Diamond$

You will be rewarded if you get the semantics (more or less) right.

Valid in our framework			
FC	$P(\alpha \vee \beta) \leftrightarrow (P\alpha \wedge P\beta)$	CD	$P(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow P((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
CE	$P(\alpha \wedge \beta) \rightarrow (P\alpha \wedge P\beta)$	DC1	$P((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \rightarrow P(\alpha \vee (\beta \wedge \gamma))$
Invalid in our framework			
CA	$(P\alpha \wedge P\beta) \rightarrow P(\alpha \wedge \beta)$	DCr	$P(\alpha \vee (\beta \wedge \gamma)) \rightarrow P((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$
RP	$P\alpha \rightarrow P(\alpha \vee \beta)$	EX	$P\alpha \rightarrow P(\alpha \wedge \alpha)$

DCr is invalid: imagine you are given a coupon that permits you to take a **hamburger** *or* a menu of **Belgian fries** *and* **salad**, this does not mean you can take a **hamburger** *or* **fries**, *and* a **hamburger** *or* **salad**.

CA and DCr are valid in Boolean-algebra-based approaches, such as [?, ?]; CD is invalid in [?]; DCr is valid in the hybrid approach based on BSMML [?]; and CE is not valid in [?].

About the “innocent” EX: $P\alpha \rightarrow P(\alpha \wedge \alpha)$

It is **not** as innocent as it seems. Under free choice and acceptable distribution, it leads to the unacceptable

$$P(\alpha \vee \beta) \rightarrow P(\alpha \wedge \beta)!$$

$$P(\alpha \vee \beta)$$

$$\implies P((\alpha \vee \beta) \wedge (\alpha \vee \beta)) \quad (\text{EX})$$

$$\iff P(((\alpha \vee \beta) \wedge \alpha) \vee ((\alpha \vee \beta) \wedge \beta)) \quad (\text{CD})$$

$$\iff P((\alpha \vee \beta) \wedge \alpha) \wedge P((\alpha \vee \beta) \wedge \beta) \quad (\text{FC})$$

$$\iff P((\alpha \wedge \alpha) \vee (\alpha \wedge \beta)) \wedge P((\beta \wedge \alpha) \vee (\beta \wedge \beta)) \quad (\text{CD, commutativity})$$

$$\iff P(\alpha \wedge \alpha) \wedge P(\alpha \wedge \beta) \wedge P(\beta \wedge \alpha) \wedge P(\beta \wedge \beta) \quad (\text{FC})$$

$$\implies P(\alpha \wedge \beta) \quad (\text{TAUT})$$

Let's get to the details.

Language and Semantics

Language AT of action types

Definition (Action Type AT)

Given a countable set of propositional letters P , the language of action types (**AT**) is defined as follows (no implication for now):

$$\alpha := p \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha)$$

where $p \in P$.

We use atomic propositional letters to represent atomic action types like “drink coffee”, “do homework”, “go to hospital”, etc. Complex action types like “eat cookies and drink coffee”, “do homework or play computer games” can also be expressed.

Definition (Language DLSP)

Given **AT**, the language of deontic logic for strong permission (**DLSP**) is defined as follows:

$$\varphi := \perp \mid p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid \neg\varphi \mid \mathbf{P}\alpha$$

where $p \in P$ and $\alpha \in \mathbf{AT}$.

We call formulas containing the deontic operator **P** *deontic formulas* and other formulas *non-deontic*.

Action space

Following the BHK-style definition:

Definition (Action Token Space)

Given P and a non-empty set I of atomic action tokens such that $I \cap \{0, 1\} = \emptyset$, an action (token) space S is a function based on I and **AT** satisfying the following constraints:

1. $S(p) \neq \emptyset \subseteq I$ for any $p \in P$;
2. $S(\alpha \wedge \beta) = S(\alpha) \times S(\beta)$;
3. $S(\alpha \vee \beta) = (S(\alpha) \times \{0\}) \cup (S(\beta) \times \{1\})$.

S is a **singleton action space** if $|S(p)| = 1$ for all $p \in P$. People may treat types and tokens alike.

For example, action tokens for a disjunctive action type $(\alpha \vee \beta)$ are the *disjoint* union of tokens of α and β .

Deontic Model

Definition (Deontic Model)

A deontic model \mathcal{M} for **DLSP** is a tuple (S, W, R, A) where S is an action space, W is a non-empty set of possible worlds, $R \subseteq W \times W$, and A is a binary function over $\mathbf{AT} \times W$ such that for any $p \in P$, $\alpha, \beta \in \mathbf{AT}$ and $w \in W$:

- $A(p, w) \subseteq S(p)$;
- $A(\alpha \wedge \beta, w) = A(\alpha, w) \times A(\beta, w)$;
- $A(\alpha \vee \beta, w) = (A(\alpha, w) \times \{0\}) \cup (A(\beta, w) \times \{1\})$;

A pointed model is a pair (\mathcal{M}, w) where w is in \mathcal{M} . A **singleton deontic model** is a model based on a singleton action space.

The function A gives each deontially good world its *executed* action tokens.

Definition (Semantics)

For any $\varphi \in \mathbf{DLSP}$ and any pointed deontic model \mathcal{M}, w where $\mathcal{M} = (S, W, R, A)$, the satisfaction relation is defined as follows:

$\mathcal{M}, w \not\models \perp$

$\mathcal{M}, w \models p \iff A(p, w) \neq \emptyset$

$\mathcal{M}, w \models (\varphi \wedge \psi) \iff \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi$

$\mathcal{M}, w \models (\varphi \vee \psi) \iff \mathcal{M}, w \models \varphi \text{ or } \mathcal{M}, w \models \psi$

$\mathcal{M}, w \models (\varphi \rightarrow \psi) \iff \mathcal{M}, w \not\models \varphi \text{ or } \mathcal{M}, w \models \psi$

$\mathcal{M}, w \models \neg\varphi \iff \mathcal{M}, w \not\models \varphi$

$\mathcal{M}, w \models \mathbf{P}\alpha \iff \text{for any } a \in S(\alpha), \text{ there is a } v \text{ s.t. } wRv \text{ and } a \in A(\alpha, v)$

We use \models_s to denote semantic consequence w.r.t. singleton deontic models. We say φ is valid (s-valid) if $\models_s \varphi$ ($\models_s \varphi$).

Recall the preview

Valid in our framework			
FC	$P(\alpha \vee \beta) \leftrightarrow (P\alpha \wedge P\beta)$	CD	$P(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow P((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
CE	$P(\alpha \wedge \beta) \rightarrow (P\alpha \wedge P\beta)$	DC1	$P((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \rightarrow P(\alpha \vee (\beta \wedge \gamma))$
Invalid in our framework (without further constraints)			
CA	$(P\alpha \wedge P\beta) \rightarrow P(\alpha \wedge \beta)$	DCr	$P(\alpha \vee (\beta \wedge \gamma)) \rightarrow P((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$
RP	$P\alpha \rightarrow P(\alpha \vee \beta)$	EX	$P\alpha \rightarrow P(\alpha \wedge \alpha)$

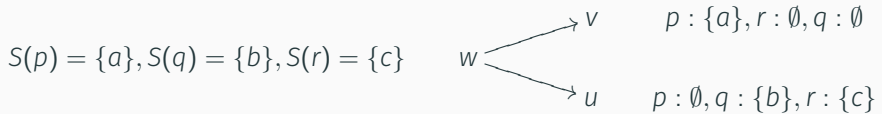
The commutativity and associativity are valid.

$$P(\alpha \wedge \beta) \leftrightarrow P(\beta \wedge \alpha) \quad P((\alpha \wedge \beta) \wedge \gamma) \leftrightarrow P(\alpha \wedge (\beta \wedge \gamma))$$

$$P(\alpha \vee \beta) \leftrightarrow P(\beta \vee \alpha) \quad P((\alpha \vee \beta) \vee \gamma) \leftrightarrow P(\alpha \vee (\beta \vee \gamma))$$

Invalidity of DCr: $\mathbf{P}(\alpha \vee (\beta \wedge \gamma)) \rightarrow \mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$

The rightmost part below demonstrates the definition of A on u, v , e.g., $A(p, v) = \{a\}$ and $A(q, u) = \{b\}$.



$S(p \vee (q \wedge r))$ contains $(a, 0)$ and $((b, c), 1)$ only, which are executable on v and u respectively, thus $\mathbf{P}(p \vee (q \wedge r))$ is true on w . However, the token $((a, 0), (c, 1))$ in $S((p \vee q) \wedge (p \vee r))$ is not executable on u nor v , thus $\mathbf{P}((p \vee q) \wedge (p \vee r))$ is false on w . Note that this model is also a singleton model so DCr is not s-valid.

A weaker version of EX over singleton models

The following formula (denoted by **EXP**) is valid with respect to the class of **singleton deontic models**:

$$\models_s \mathbf{P}(p_1 \wedge \dots \wedge p_k) \rightarrow \mathbf{P}(m_1 \cdot p_1 \wedge \dots \wedge m_k \cdot p_k),$$

where $p_1, \dots, p_k \in P$ are pairwise distinct, $k, m_i \in \mathbb{N}_{>0}$ for any $1 \leq i \leq k$. Here $m_i \cdot p_i$ represents the conjunction of m_i copies of p_i .

Proof Systems

System DLSP

Axioms

(TAUT) Propositional Tautologies

(FC) $\mathbf{P}(\alpha \vee \beta) \leftrightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$

(CE) $\mathbf{P}(\alpha \wedge \beta) \rightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$

(COM $_{\wedge}$) $\mathbf{P}(\alpha \wedge \beta) \leftrightarrow \mathbf{P}(\beta \wedge \alpha)$

(ASSO $_{\wedge}$) $\mathbf{P}((\alpha \wedge \beta) \wedge \gamma) \leftrightarrow \mathbf{P}(\alpha \wedge (\beta \wedge \gamma))$

(CD) $\mathbf{P}(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow \mathbf{P}((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$

Rules

(MP) Given φ and $(\varphi \rightarrow \psi)$, infer ψ .

System DLSP^S

System DLSP with the following axiom

(EXP) $\mathbf{P}(p_1 \wedge \dots \wedge p_k) \rightarrow \mathbf{P}(m_1 \cdot p_1 \wedge \dots \wedge m_k \cdot p_k)$

Normal form

We use DLSP to rewrite a DLSP -formula into a conjunction of formulas in the shape of $\mathbf{P}(p_1 \wedge \dots \wedge p_n)$.

$$\mathbf{P}(p_1 \vee (p_2 \wedge ((p_3 \vee p_4) \wedge p_5))).$$

The formula is logically equivalent to

1. $\mathbf{P}p_1 \wedge \mathbf{P}(p_2 \wedge ((p_3 \vee p_4) \wedge p_5))$ (FC)
2. $\mathbf{P}p_1 \wedge \mathbf{P}((p_5 \wedge p_2) \wedge (p_3 \vee p_4))$ ($\text{ASSO}_\wedge + \text{COM}_\wedge$)
3. $\mathbf{P}p_1 \wedge \mathbf{P}(((p_5 \wedge p_2) \wedge p_3) \vee ((p_5 \wedge p_2) \wedge p_4))$ (CD)
4. $\mathbf{P}p_1 \wedge \mathbf{P}((p_5 \wedge p_2) \wedge p_3) \wedge \mathbf{P}((p_5 \wedge p_2) \wedge p_4)$ (FC)

Lemma (Normal Form for $\mathbf{P}\alpha$)

For any $\alpha \in \text{AT}$, $\mathbf{P}\alpha$ is logically equivalent to a formula of the form $(\mathbf{P}\beta_1 \wedge \dots \wedge \mathbf{P}\beta_k)$ where for each $1 \leq i \leq k$, β_i is in the shape of $\mathbf{P}(p_1 \wedge \dots \wedge p_n)$, which is called a **normal form** for $\mathbf{P}\alpha$.

Normal form

For any formula $\varphi \in \mathbf{DLSP}$, φ is logically equivalent to a formula in the following language (denoted by \mathbf{DLSP}^*):

$$\psi ::= \perp \mid p \mid \mathbf{P}(p_1 \wedge \dots \wedge p_n) \mid \neg\psi \mid (\psi \wedge \psi) \mid (\psi \vee \psi) \mid (\psi \rightarrow \psi),$$

where $p, p_1, \dots, p_n \in P$.

To show the completeness, we will construct for each consistent set of formulas a model.

Completeness

All-distinct action token

Note that due to the validity of ASSO_\wedge and COM_\wedge , we will treat an action token of type $(p_1 \wedge \dots \wedge p_n)$ as an n -ary tuple of action tokens modulo paring.

Definition (All-Distinct Token)

An action token of type $(p_1 \wedge \dots \wedge p_n)$ is *all-distinct* if tokens of the same atomic action type in the tuple are pairwise distinct.

We need to first build the **action spaces** before constructing the canonical model.

Canonical action space

Now let Σ be a maximally DLSP-consistent set of DLSP* formulas.

Definition (Canonical Action Space)

Given Σ , we define S_{Σ}^C by distinguishing the two cases of $p \in P$:

- If there is an $i \in \mathbb{N}_{>0}$ such that the formula $\neg \mathbf{P}(i \cdot p) \in \Sigma$, assume that n is the least and let $S_{\Sigma}^C(p) := \{p^1, p^2, \dots, p^n\}$, in which each p^i is the propositional letter p superscript with the numeral i .
- If not, i.e., $\mathbf{P}(i \cdot p) \in \Sigma$ for all $i \in \mathbb{N}_{>0}$, let $S_{\Sigma}^C(p) := \{p^1, p^2, \dots\}$.

For any composite $\alpha \in \mathbf{AT}$, we define $S_{\Sigma}^C(\alpha)$ recursively as in the definition of S .

Note that for distinct $p, q \in P$, $S_{\Sigma}^C(p) \cap S_{\Sigma}^C(q) = \emptyset$.

Existence of all-distinct token

Lemma

For any formula φ of the form $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \dots \wedge m_k \cdot p_{t_k})$ where p_{t_i}, p_{t_j} are pairwise distinct, if $\varphi \in \Sigma$, then for any $1 \leq j \leq k$, $m_j < |S_\Sigma^C(p_{t_j})|$.

Proof.

Prove by contradiction. □

This lemma shows the size of the action space is *more than enough* to guarantee the existence of all-distinct action tokens of the type $(m_1 \cdot p_{t_1} \wedge \dots \wedge m_k \cdot p_{t_k})$ when $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \dots \wedge m_k \cdot p_{t_k}) \in \Sigma$.

The very idea of canonical model

Based on the lemma and S_{Σ}^C , we will build a pointed deontic model $\mathcal{M}_{\Sigma}^C, w$ such that the truth lemma holds.

The idea is simple: given a designated world w , build the accessible worlds according to formulas

$$\mathbf{P}(m_1 \cdot p_{t_1} \wedge \dots \wedge m_k \cdot p_{t_k}) \in \Sigma.$$

The subtlety is that we should only realize action tokens that are *necessary* to witness the truth of those φ , but no more, for we also need tokens not realizable to witness $\neg\mathbf{P}(p_1 \wedge \dots \wedge p_n) \in \Sigma$. The later task is doable because we have some spare tokens in S_{Σ}^C based on the previous Lemma.

Functional representation

Fixing an ordering of propositional letters p_0, p_1, p_2, \dots , we only need to consider $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \dots \wedge m_k \cdot p_{t_k}) \in \Sigma$ such that p_{t_i} and p_{t_j} are distinct and ordered, e.g., $\mathbf{P}(3 \cdot p_2 \wedge 4 \cdot p_6)$.

Definition

For any φ of the form $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \dots \wedge m_k \cdot p_{t_k}) \in \Sigma$ such that p_{t_i} and p_{t_j} are distinct and ordered according to the order of propositional letters, we define $f_\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$f_\varphi(i) = \begin{cases} m_j & i = t_j \text{ for some } 1 \leq j \leq k; \\ 0 & i \neq t_j \text{ for any } 1 \leq j \leq k. \end{cases}$$

For example, $\mathbf{P}(3 \cdot p_2 \wedge 4 \cdot p_6)$ is represented by the function f such that $f(2) = 3$, $f(6) = 4$ and $f(i) = 0$ for any $i \in \mathbb{N} \setminus \{2, 6\}$. We collect these (countably many) functions in F_Σ .

Functional representation

Definition

For any $f \in F_\Sigma$, we define $G_f := \{g : \mathbb{N} \rightarrow \mathcal{P}(\bigcup_{p \in P} (S_\Sigma^C(p))) \mid$
for any $i \in \mathbb{N}$, $g(i) \subseteq S_\Sigma^C(p_i)$ and $|g(i)| = f(i)\}$.

Intuitively, each $g \in G_f$ assigns a subset of the canonical action space of each p_i whose cardinality is $f(i)$. It follows if $f(i) = 0$ then $g(i) = \emptyset$. In fact, each $g \in G_f$ can be treated as an *all-distinct* token of the type in φ . Let $G_\Sigma = \bigcup \{G_f \mid f \in F_\Sigma\}$.

Proposition

Given a MCS Σ and any distinct $f, f' \in F_\Sigma$, we have: (1) G_f is not empty; (2) $G_f \cap G_{f'} = \emptyset$.

Canonical deontic model

Definition (Canonical Deontic Model)

Given a MCS Σ , we define the model $\mathcal{M}_\Sigma^C = (S_\Sigma^C, W^C, R^C, A^C)$ where:

- $W^C = \{w\} \cup G_\Sigma$; $R^C = \{(w, g) \mid g \in G_\Sigma\}$;
- $A^C(p_i, u) = \begin{cases} S_\Sigma^C(p_i) & \text{if } u = w \text{ and } p_i \in \Sigma, \\ \emptyset & \text{if } u = w \text{ and } p_i \notin \Sigma, \\ u(i) & \text{if } u \in G_\Sigma; \end{cases}$

and $A^C(\alpha, u)$ for composite α is defined as in definition of deontic model.

If $g \in G_\Sigma$ then there is a unique $f \in F_\Sigma$ s.t. $g \in G_f$. Intuitively, each $g \in G_f$ realizes some all-distinct token of the formula $\mathbf{P}\alpha \in \Sigma$ corresponding to f , and G_f realize all the necessary tokens.

Lemma (Truth Lemma for DLSP)

Given a MCS Σ . For any $\varphi \in \Sigma$,

$$\mathcal{M}_{\Sigma}^C, w \models \varphi \iff \varphi \in \Sigma.$$

Proof.

Prove by induction on the structure of φ . We only show the inductive case when $\varphi = \mathbf{P}(p_1 \wedge \dots \wedge p_n)$. By (COM_{\wedge}) and (ASSO_{\wedge}) , φ is logically equivalent to a formula ψ of the form $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \dots \wedge m_k \cdot p_{t_k})$ where p_{t_i} and p_{t_j} are pairwise distinct and ordered.

Proof.

\Leftarrow : Assume that $\psi \in \Sigma$. We have the corresponding $f_\psi \in F_\Sigma$. Then each all-distinct token of type in ψ is represented and thus realized by some $g \in G_{f_\psi}$. And this will guarantee all tokens be realized.

\Rightarrow : Assume that $\psi \notin \Sigma$. To show $\mathcal{M}_\Sigma^C, w \not\models \psi$, we need to find some token in $S_\Sigma^C(m_1 \cdot p_{t_1} \wedge \dots \wedge m_k \cdot p_{t_k})$ cannot be witnessed by any successor. The crucial point here is that our definition of S_Σ^C and A^C together guarantee that some action tokens are indeed **left out** at every $g \in G_\Sigma$.

Proof.

Now we consider two cases:

- If for any $1 \leq j \leq k$, $m_j \leq |S_{\Sigma}^C(p_{t_j})|$, we take an all-distinct token $x \in S(m_1 \cdot p_{t_1} \wedge \dots \wedge m_k \cdot p_{t_k})$ and show it is not realizable in G_{Σ} , thus $\mathcal{M}_{\Sigma}^C, w \not\models \psi$. Suppose not, so there is a $g \in G_{\Sigma}$ that realizes x , then there is a unique f such that $g \in G_f$. Since g realizes all-distinct token x , then we have $f(t_j) = |g(t_j)| = |A^C(p_{t_j}, g)| \geq m_j$ for any $1 \leq j \leq k$. Due to our construction, there must be a $\chi \in \Sigma$ such that $f = f_{\chi}$.

Therefore, χ must be of the form

$$\mathbf{P}((m'_1 \cdot p_{t_1} \wedge \dots \wedge m'_k \cdot p_{t_k}) \wedge (m'_{k+1} \cdot p_{t_{k+1}} \wedge \dots \wedge m'_{k+l} \cdot p_{t_{k+l}})) \in \Sigma$$
 such that $m'_j = f(t_j) \geq m_j$. By (CE) and (MP), $\psi \in \Sigma$, contradicting to the assumption that $\psi \notin \Sigma$.

Proof.

- If there is $1 \leq j \leq k$ such that $m_j > |S_{\Sigma}^C(p_{t_j})|$, thus $S_{\Sigma}^C(p_{t_j})$ is finite, say $|S_{\Sigma}^C(p_{t_j})| = n$. Suppose towards a contradiction that $\mathcal{M}_{\Sigma}^C, w \models \psi$. Thus by the validity of **CE**, $\mathcal{M}_{\Sigma}^C, w \models \mathbf{P}(n \cdot p_{t_j})$. Hence, to realize the token using all the atomic tokens in $S_{\Sigma}^C(p_{t_j})$, there must be a $g \in G$ such that $A^C(p_{t_j}, g) = g(t_j) = S_{\Sigma}^C(p_{t_j})$. Further there must be a unique f such that $g \in G_f$ and $f(t_j) = |g(t_j)| = |S_{\Sigma}^C(p_{t_j})| = n$. Therefore there is a $\chi \in \Sigma$ such that $f = f_{\chi}$. However this means χ must be in the shape of $\mathbf{P}(n \cdot p_{t_j} \wedge \beta) \in \Sigma$. By **(CE)**, $\mathbf{P}(n \cdot p_{t_j}) \in \Sigma$ contradicting to the fact that $|S_{\Sigma}^C(p_{t_j})| = n$. Therefore $\mathcal{M}_{\Sigma}^C, w \not\models \psi$.

□

Completeness Theorem for DLSP

Based on the truth lemma, by a Lindenbaum-like argument, we can show:

Theorem (Completeness Theorem for DLSP)

DLSP is strongly complete with respect to the class of all deontic models.

Note that DLSP is also complete over all *serial* models, i.e., the models where every node has a successor.

Canonical singleton action space

Now we prove completeness theorem for \mathbb{DLSP}^S . Let Σ be a maximally \mathbb{DLSP}^S -consistent set of \mathbf{DLSP}^* formulas.

Definition (Canonical Singleton Action Space)

Given Σ , we define the canonical singleton action space S_Σ^S such that $S_\Sigma^S(p) := \{p\}$ for any $p \in P$ and $S_\Sigma^S(\alpha)$ is defined recursively as above for any composite $\alpha \in \mathbf{AT}$.

To define the canonical singleton deontic model $\mathcal{M}_\Sigma^S = (S_\Sigma^S, W^S, R^S, A^S)$, still fixing an ordering of propositional letters in advance, we essentially apply the same method as before.

However, our definition will be simplified here. Since $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \dots \wedge m_k \cdot p_{t_k})$ is logically equivalent to $\mathbf{P}(p_{t_1} \wedge \dots \wedge p_{t_k})$ by extra validities (**EXP**) in \mathbf{DLSP}^S , we only consider formulas φ of the latter form in Σ and define f_φ, G_{f_φ} as before. So, for any $i \in \mathbb{N}$, if $i = t_j$, then $f_\varphi(i) = 1$, and otherwise $f_\varphi(i) = 0$. And by this feature of f_φ and S_Σ^S as singleton action space, there is indeed a unique $g \in G_{f_\varphi}$ such that $g(i) = \{p_{t_j}\}$ if $i = t_j$ and $g(i) = \emptyset$ otherwise. We collect all such g in G'_Σ .

Canonical singleton deontic model

Definition (Canonical Singleton Deontic Model)

Given Σ , we define the singleton deontic model

$\mathcal{M}_{\Sigma}^S = (S_{\Sigma}^S, W^S, R^S, A^S)$ where:

- $W^S = \{v\} \cup G'_{\Sigma}$; $R^S = \{(v, g) \mid g \in G'_{\Sigma}\}$;
- $A^S(p_i, u) = \begin{cases} S_{\Sigma}^S(p_i) & \text{if } u = v \text{ and } p_i \in \Sigma, \\ \emptyset & \text{if } u = v \text{ and } p_i \notin \Sigma, \\ u(i) & \text{if } u \in G'_{\Sigma}; \end{cases}$

and $A^S(\alpha, u)$ for composite α is defined as in Definition of deontic model.

Completeness theorem for DLSP^s

Lemma (Truth Lemma for DLSP^s)

Let Σ be a maximally DLSP^s -consistent set of DLSP^* formulas.

For any $\varphi \in \Sigma$,

$$\mathcal{M}_{\Sigma}^s, v \vDash_s \varphi \iff \varphi \in \Sigma.$$

Theorem (Completeness Theorem for DLSP^s)

DLSP^s is strongly complete with respect to the class of all singleton deontic models.

Extensions

Higher-order permission

Giving a permission itself can also be an action type!

Then we can express $\mathbf{PP}p$, $\neg\mathbf{P}(p \vee \mathbf{P}q)$, $\mathbf{PP}p \rightarrow \mathbf{P}p$, $\mathbf{P}p \wedge \neg\mathbf{PP}p \dots$

We can give the interpretation for $\mathbf{P}\alpha$ as a type.

- $S(\mathbf{P}\alpha) = \{c_\alpha\}$.
- $A(\mathbf{P}\alpha, w) = \begin{cases} \{c_\alpha\} & w \models \mathbf{P}\alpha \\ \emptyset & \text{otherwise} \end{cases}$

The system \mathbf{DLSP} with the following rule is sound and complete:

$$\boxed{\text{Given } \vdash \bigwedge \overline{\mathbf{P}\alpha} \rightarrow \bigwedge \overline{\mathbf{P}\beta}, \text{ infer } \vdash \chi \rightarrow \chi[\bigwedge \overline{\mathbf{P}\beta} / \bigwedge \overline{\mathbf{P}\alpha}].}$$

With an extra axiom $\mathbf{PP}\alpha \rightarrow \mathbf{P}\alpha$, the logic is complete over transitive frames.

Simultaneous conjunction

In DLSP, (CE) : $P(\alpha \wedge \beta) \rightarrow (P\alpha \wedge P\beta)$ is **valid**. On the other hand, it also seems controversial, e.g. [?]:

*Flight safety: in case of emergency, you are permitted to wear the parachute **and** jump out of the plane; but you are **not** permitted to jump directly.*

- Here, the conjunctive action is **simultaneous** in the sense that both conjuncts need to be *uniformly* executed by one and the same token.
- For example, sometimes you are required to take different medicines **together** to guarantee they all work.
- We can express such simultaneous action types by alternatively interpreting conjunction as **intersection** rather than product.

Simultaneous action space and model

Given P , we still work in language \mathbf{AT}_P and \mathbf{DLSP}_P .

Definition (Simultaneous Action Token Space)

Given a non-empty set I of atomic action tokens, a simultaneous action (token) space S is a function with domain \mathbf{AT}_P such that:

1. $S(p) \neq \emptyset \subseteq I$ for any $p \in P$;
2. $S(\alpha \vee \beta) = S(\alpha) \cup S(\beta)$ and $S(\alpha \wedge \beta) = S(\alpha) \cap S(\beta)$.

Definition (Simultaneous Deontic Model)

A simultaneous deontic model \mathcal{M} is a tuple (S, W, R, A) where S is a simultaneous action space, W, R are as before, A matches S and satisfies **coinstantiation** that for any $\alpha, \beta \in \mathbf{AT}$ and $w \in W$,

$$\text{if } x \in S(\alpha) \cap S(\beta), \text{ then } x \in A(\alpha, w) \iff x \in A(\beta, w).$$

System $DLSP^{SC}$ and completeness

Axioms

(TAUT) Propositional Tautologies

(FC) $P(\alpha \vee \beta) \leftrightarrow (P\alpha \wedge P\beta)$

(AB) $P(\alpha \wedge \alpha) \rightarrow P\alpha$

(CI) $P\alpha \rightarrow P(\alpha \wedge \beta)$

(COM $_{\wedge}$) $P(\alpha \wedge \beta) \leftrightarrow P(\beta \wedge \alpha)$

(ASSO $_{\wedge}$) $P((\alpha \wedge \beta) \wedge \gamma) \leftrightarrow P(\alpha \wedge (\beta \wedge \gamma))$

(CD) $P(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow P((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$

Rules

(MP) Given φ and $(\varphi \rightarrow \psi)$, infer ψ .

Theorem (Soundness and Completeness Theorem for $DLSP^{SC}$)

$DLSP^{SC}$ is sound and strongly complete with respect to the class of all simultaneous deontic models.

Conclusions

Conclusions

- We formalize strong permission as a $\forall x\Diamond$ bundle
- The propositional formulas are action types whose tokens are given by a BHK-style recursive definition
- The resulting logic admits FC and most other good properties, if not all.
- It also predicts phenomena aligned with our linguistic intuition, which were not discussed in the literature

Inspired by the BHK interpretation:

- Adding \neg in the scope of modalities
- Capturing various versions of **O** and **F**
- Adding implications in the scope of modalities
- Adding sequential “and” such that the tokens are sequences
- Try to solve more puzzles!