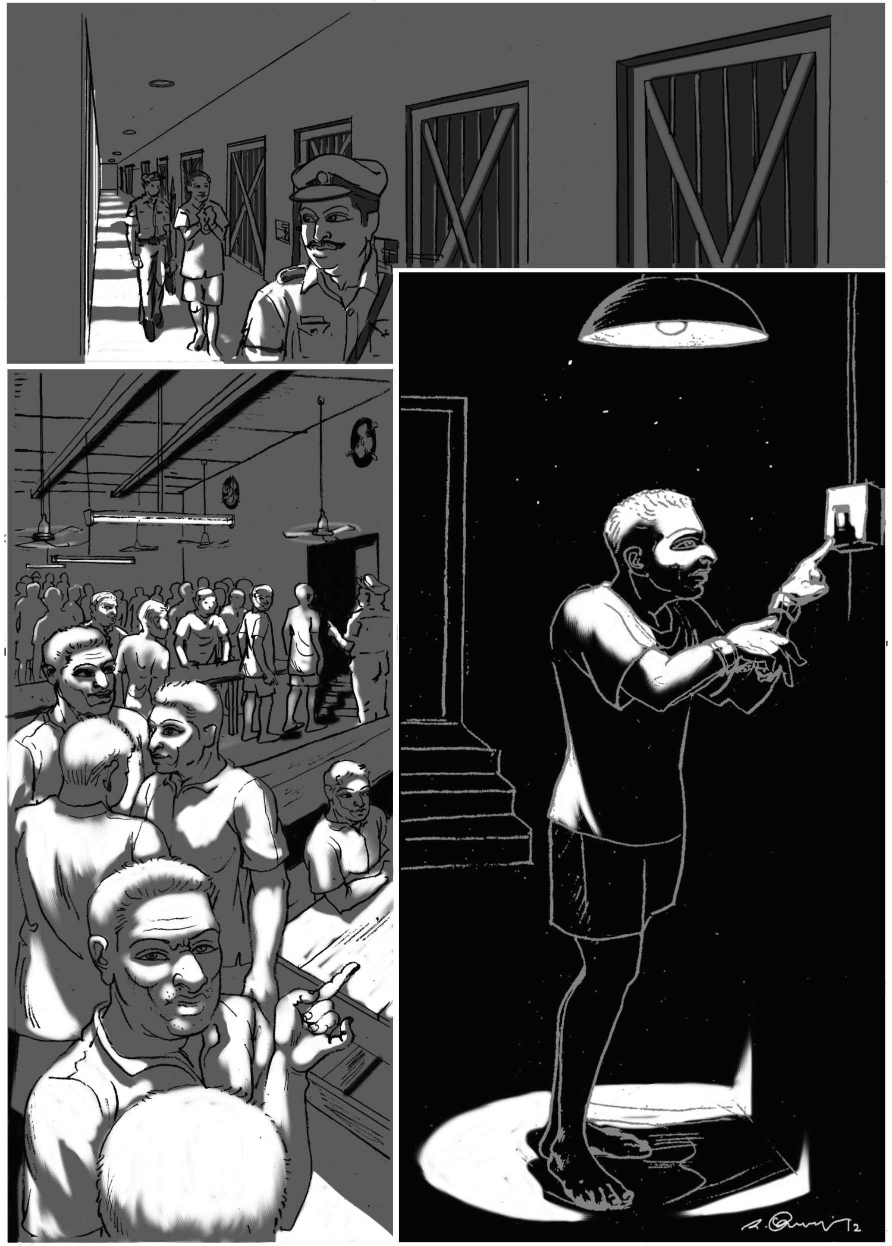
One hundred prisoners and a light bulb



One hundred prisoners and a light bulb

A group of 100 prisoners, all together in the prison dining area, are told that they will be all put in isolation cells and then will be interrogated one by one in a room containing a light with an on/off switch. The prisoners may communicate with one another by toggling the light-switch (and that is the only way in which they can communicate). The light is initially switched off. There is no fixed order of interrogation, or interval between interrogations, and at any stage the same prisoner will later be interrogated again. When interrogated, a prisoner can either do nothing, or toggle the light-switch, or announce that all prisoners have been interrogated. If that announcement is true, the prisoners will (all) be set free, but if it is false, they will all be executed. While still in the dining room, and before the prisoners go to their isolation cells (forever), can the prisoners agree on a protocol that will set them free?

100 prisoners — not a solution

Let there be one prisoner:

Protocol: *If a prisoner enters the interrogation room, he announces that all prisoners have been interrogated.*

Let there be two prisoners:

Protocol: *If a prisoner enters the interrogation room and the light is off, he turns it on; if a prisoner enters the interrogation room and the light is on, and he has not turned on the light at a previous interrogation, he announces that all prisoners have been interrogated.*

Let there be three prisoners:

Protocol: ...

100 prisoners — solution Protocol for $n \geq 3$ prisoners

The n prisoners appoint one amongst them as the counter. The non-counting prisoners are the followers. The followers follow the following protocol: the first time they enter the room when the light is off, they turn it on; on all other occasions, they do nothing. The counter follows a different protocol. When the light is on when he enters the interrogation room, he turns it off. When he turns off the light for the $(n-1)$st time, he announces that everybody has been interrogated.

Protocol for $n \geq 3$ prisoners

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Let us picture a number of executions of this protocol for $n = 3$. The upper index: state of the light. The lower index: the number of times the light has been turned off. Anne is the counter.

— ${}^0\text{Bob}{}^1\text{Anne}^0_1\text{Caro}{}^1\text{Anne}^0_2$
— ${}^0\text{Anne}{}^0\text{Bob}{}^1\text{Caro}{}^1\text{Anne}^0_1\text{Bob}{}^0\text{Anne}^0_1\text{Caro}{}^1\text{Caro}{}^1\text{Bob}{}^1\text{Bob}{}^1\text{Anne}^0_2$
— ${}^0\text{Bob}{}^1\text{Anne}^0_1\text{Bob}{}^0\text{Caro}{}^1\text{Bob}{}^1\text{Anne}^0_2$

If the scheduling is fair, then the protocol will terminate.

Followers can also count

A follower may know before the counter that everybody has been interrogated. E.g., follower Bob may know it before counter Anne:

— $^{0}\text{Bob}^{1}\text{Anne}_{1}^{0}\text{Bob}^{0}\text{Caro}^{1}\textbf{Bob}^{1}\text{Anne}_{2}^{0}$

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Assume a single interrogation per day takes place.
When can the prisoners expect to be set free from prison?

When will you get out of prison?

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When can the prisoners expect to be set free from prison?

non-counter / counter / another non-counter / counter / etc.

$\frac{99}{100}$ / $\frac{1}{100}$ / $\frac{98}{100}$ / $\frac{1}{100}$ / etc.

$\frac{100}{99}$ / $\frac{100}{1}$ / $\frac{100}{98}$ / $\frac{100}{1}$ / etc.

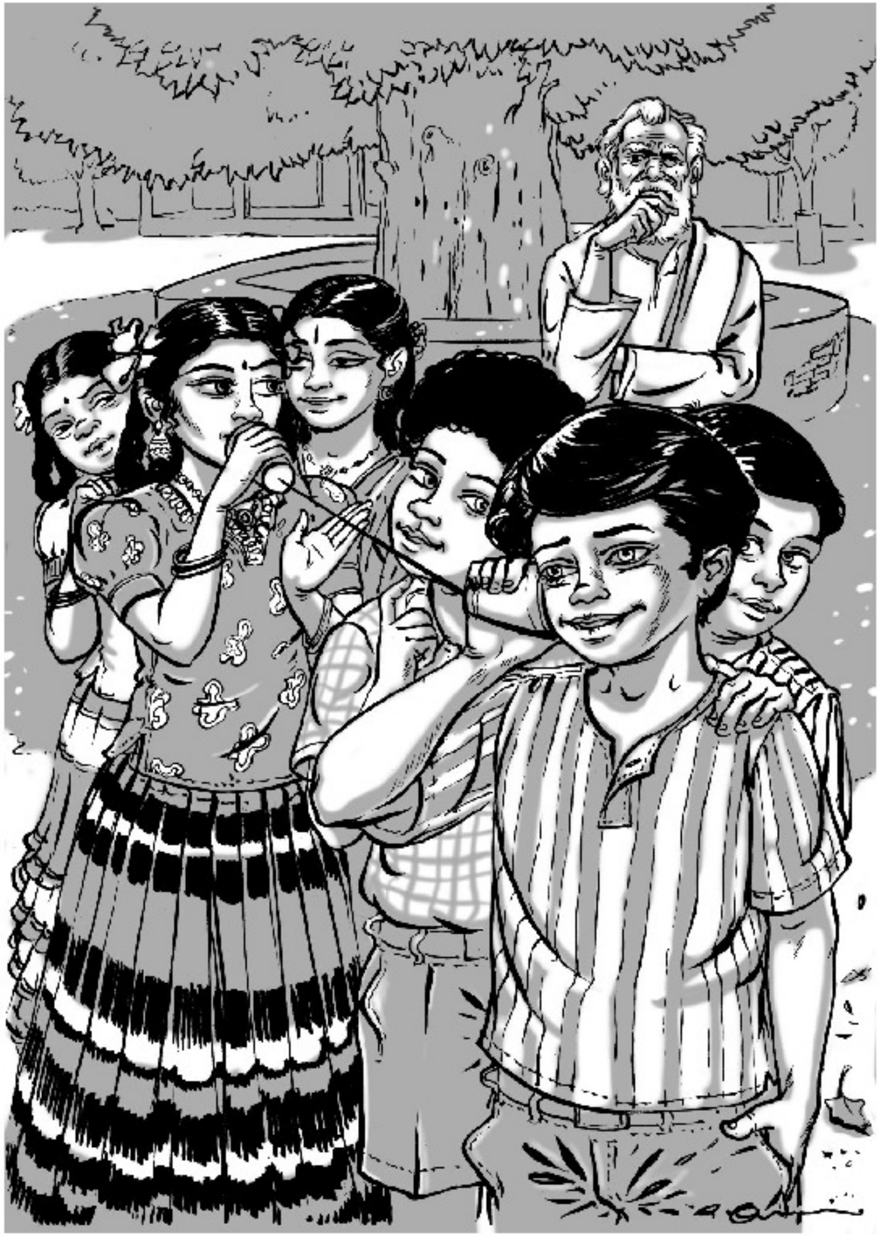
Summation:

$$\sum_{i=1}^{99}\left(\frac{100}{i}+\frac{100}{1}\right) = 99\cdot 100+100\cdot\sum_{i=1}^{99}\frac{1}{i} = 9,900+518 \text{ days } \approx 28.5 \text{ years}$$

This can be reduced to around 9 years. The minimum is unknown.

Relation to **Coupon Collector's Problem** (De Moivre):
$n \cdot \sum_{i=1}^{n} \frac{1}{i} = n \cdot H_n$ where H_n is the harmonic number.

Gossip



Gossip: agents exchanging secrets

Six friends each know a secret. They can call each other. In each call they exchange all the secrets they know. How many calls are needed for everyone to know all secrets?

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First consider four friends a, b, c, d who hold secrets A, B, C, D.
Four calls ab; cd; ac; bd distribute all secrets.

$$A.B.C.D \overset{ab}{\rightarrow} AB.AB.C.D \overset{cd}{\rightarrow} AB.AB.CD.CD \overset{ac}{\rightarrow} ABCD.AB.ABCD.CD \overset{bd}{\rightarrow} ABCD.ABCD.ABCD.ABCD$$

Now consider friends a, b, c, d, e, f with secrets A, B, C, D, E, F.
Eight calls ae; af; ab; cd; ac; bd; ae; af distribute all secrets.

[Peer-to-peer communication; epidemiology; semantic web; ...]

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$$ABCD.AB.ABCD.CD \stackrel{bd}{\rightarrow} ABCD.ABCD.ABCD.ABCD$$

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[Peer-to-peer communication; epidemiology; semantic web; . . .]

cd : How does c know that she should call d, and not a or b?
We need knowledge-based gossip protocols.

One Hundred Prisoners and a Light Bulb: the book

authors: Hans van Ditmarsch and Barteld Kooi
illustrations: Elancheziyan

More information:
`http://personal.us.es/hvd/lightbulb.html`

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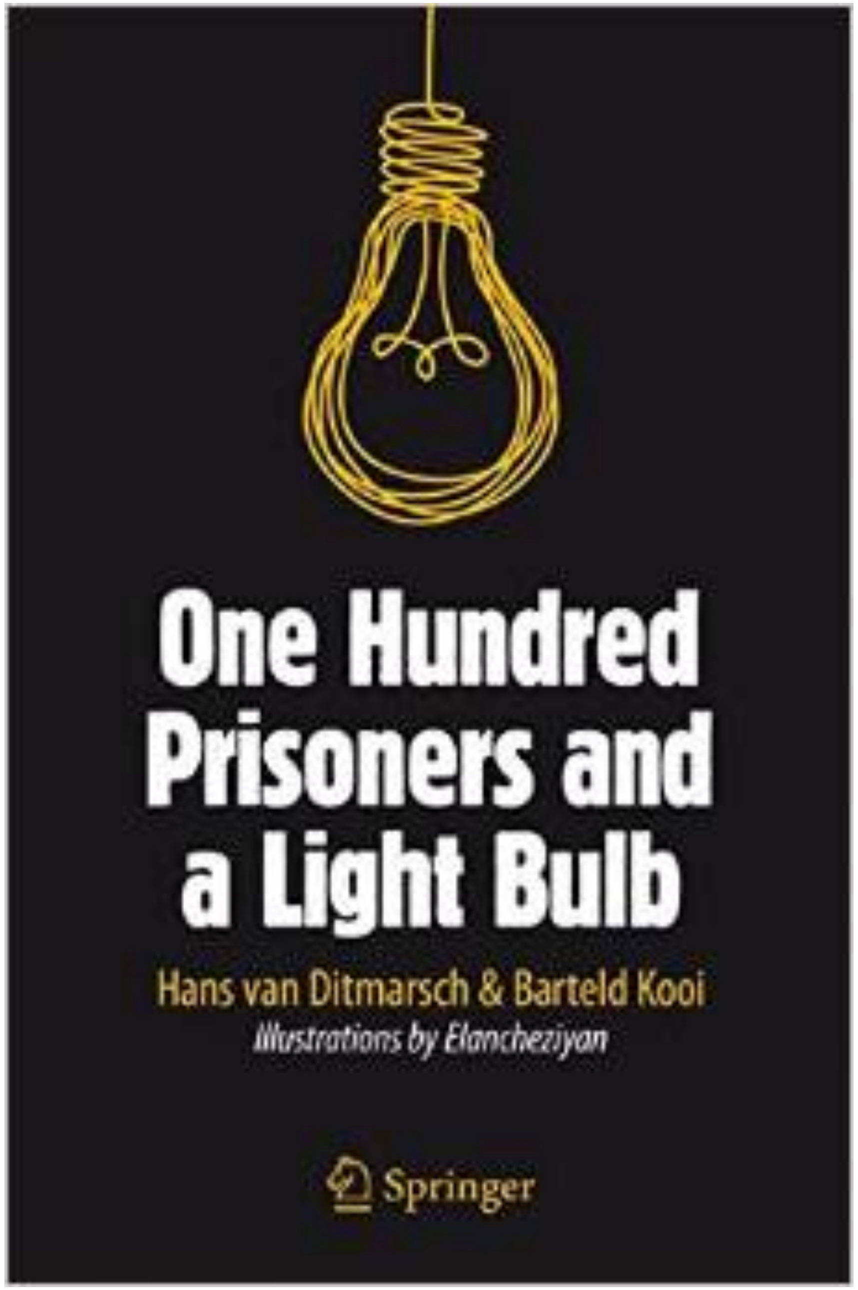

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