Bundled fragments of first-order modal logic

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Recap

$\exists\Box$-fragment as an example

Recap

The logic tool for know-wh

knowledge-that	—	propositional modal logic
knowledge-wh	—	quantified modal logic

We proposed and studied various concrete logics of know-wh by using bundles.

We often have decidable logics with low complexity. Whether this can be explained in a more theoretical term?

Disadvantages of those concrete logics

'Disadvantages' from a linguistic point of view:

- Compositionality
- Uniformity
- Expressivity

Disadvantages in terms of knowledge representation:

- Propositional epistemic logic is not really about the *content* of knowledge!

Towards a general bundled framework

What we are after:

- Expressive enough: covering the essence of those non-standard epistemic logics
- Not too much: sharing most good properties of propositional modal logic

Uniformity, compositionality, expressvity, computability: we want a predicate modal framework like the propositional modal logic.

We will give an example.

$\exists\Box$-fragment as an example

A logic framework of mention-some [Wang TARK17]

Definition ($\exists\Box$-fragment)

Given set of variables X and set of predicate symbols Ps,

$$\varphi ::= P\overline{x} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x\Box\varphi$$

where $x, y \in \mathrm{X}$, $P \in \mathrm{Ps}$.

In epistemic context $\exists x\Box\varphi$ says 'I know some x such that $\varphi(x)$'.

$\Box\varphi$ is expressible by $\exists x\Box\varphi$ if x does not occur free in φ.

We can add the equality symbol, function symbols, and constants (but it will change the computational properties).

Expressivity

- Knowing-wh: $\exists x \Box \varphi(x)$
- "I know a theorem of which I do not know any proof": $\exists x \Box \neg \exists y \Box Prove(y, x)$
- "*a* knows a country which *b* knows its capital": $\exists x \Box_a \exists y \Box_b Capital(y, x)$

First-order Kripke semantics

Definition (First-order Kripke Model)

An *increasing domain* model $\mathcal{M} = \langle W, D, \delta, R, \rho \rangle$ where:

- W is a non-empty set.
- D is a non-empty set.
- $R \in 2^{W \times W}$ is a binary relation over W.
- $\delta : W \to 2^D$ assigns to each $w \in W$ a *non-empty* local domain s.t. wRv implies $\delta(w) \subseteq \delta(v)$ for any $w, v \in W$.
- $\rho : \mathrm{Ps} \times W \to \bigcup_{n \in \omega} 2^{D^n}$ such that ρ assigns each n-ary predicate on each world an n-ary relation on D.

We write $D_w^{\mathcal{M}}$ for the local domain $\delta(w)$ in $\mathcal{M}$. If $\delta(w) = \delta(w')$ for all w, w' then it is called a *constant domain* model.

Semantics

Definition (∃□ Semantics)

$$\mathcal{M}, w, \sigma \models \exists x \Box \varphi \quad \Leftrightarrow \quad \text{there exists an } a \in D_w^{\mathcal{M}} \text{ such that } \mathcal{M}, v, \sigma[x \mapsto a] \models \varphi \text{ for all } v \text{ s.t. } wRv$$

$$\Leftrightarrow \quad \text{there exists an } a \in D_w^{\mathcal{M}} \text{ such that } \mathcal{M}, w, \sigma[x \mapsto a] \models \Box \varphi$$

∃□ fragment is indeed an extension of ML:

$\models \Box\varphi \leftrightarrow \exists x \Box \varphi$ (given x does not appear free in φ).

A formula φ is *satisfiable* if there is an increasing domain pointed model $\mathcal{M}, w$ and an assignment σ such that $\mathcal{M}, w, \sigma \models \varphi$ and $\sigma(x) \in D_w^{\mathcal{M}}$ for all $x \in X$.

$\exists\Box$-Bisimulation (inspired by monotonic and obj-world bis)

Given $\mathcal{M}$ and $\mathcal{N}$, non-empty $Z \subseteq (W_\mathcal{M} \times D^*_\mathcal{M}) \times (W_\mathcal{N} \times D^*_\mathcal{N})$ is call an $\exists\Box$-*bisimulation*, if for every $((w, \overline{a}), (v, \overline{b})) \in Z$ such that $|\overline{a}| = |\overline{b}|$ the following holds (we write $w\overline{a}$ for $(w, \overline{a})$):

PISO $\overline{a}$ and $\overline{b}$ form a "partial isomorphism" based on the interpretations of predicates at w and v respectively.

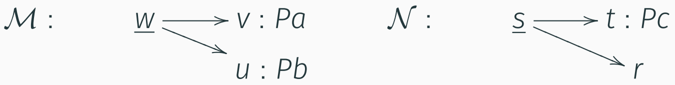
$\exists\Box$Zig For any $c \in D^\mathcal{M}_w$, there is a $d \in D^\mathcal{N}_v$ such that for any $v' \in W_\mathcal{N}$ if vRv' then there exists w' in $W_\mathcal{M}$ such that wRw' and $w'\overline{a}cZv'\overline{b}d$. $(\forall^{object}_\mathcal{M} \exists^{object}_\mathcal{N} \forall^{world}_\mathcal{N} \exists^{world}_\mathcal{M})$

$\exists\Box$Zag Symmetric to $\exists\Box$Zig.

We say $\mathcal{M}, w\overline{a}$ and $\mathcal{N}, v\overline{b}$ are $\exists\Box$-*bisimilar* ($\mathcal{M}, w\overline{a} \underline{\leftrightarrow}_{\exists\Box} \mathcal{N}, v\overline{b}$) if $|a| = |b|$ and there is an $\exists\Box$-bisimulation linking $w\overline{a}$ and $v\overline{b}$.
If there is equality symbol then PISO should respect *identity*.

Example

Consider the constant domain models $\mathcal{M}$ and $\mathcal{N}$:



where $D^{\mathcal{M}} = \{a, b\}$, $D^{\mathcal{N}} = \{c\}$. Suppose P is the only unary predicate, we can show that $\mathcal{M}, w \leftrightarroweq_{\exists\Box} \mathcal{N}, s$ by an $\exists\Box$-bisimulation Z:

$$\{(w, s), (va, tc), (ub, tc), (vb, rc), (ua, rc)\}$$

Note that $\exists\Box$Zig and $\exists\Box$Zag hold trivially for $w\bar{a}$ and $v\bar{b}$ if w and v *do not* have any successor, based on the fact that local domains are non-empty by definition.

Limited expressive power

Theorem

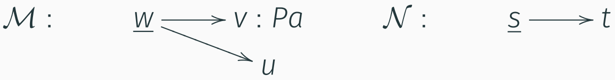
$\mathcal{M}, w\bar{a} \leftrightarroows_{\exists\Box} \mathcal{N}, v\bar{b}$ *then* $\mathcal{M}, w\bar{a} \equiv_{\mathsf{MLMS}^{\approx}} \mathcal{N}, v\bar{b}$.

Proposition

$\Box\exists xPx$, $\exists x\Diamond Px$ and $\Diamond\exists xPx$ are not *expressible in the $\exists\Box$-fragment.*

For the undefinability of $\Box\exists Px$ see the previous example.

For $\exists x\Diamond Px$, and $\Diamond\exists xPx$, consider:



where $D^{\mathcal{M}} = \{a, b\}$, $D^{\mathcal{N}} = \{c\}$ as before.

A model $\mathcal{M}$ is said to be $\exists\Box$-*saturated*, if for any $w \in W^{\mathcal{M}}$, and any finite sequence $\overline{a} \in D^*_{\mathcal{M}}$:

$\exists\Box$-type If for each finite subset Δ of a set $\Gamma(\overline{y}x)$ where $|\overline{y}| = |\overline{a}|$, $\mathcal{M}, w \vDash \exists x\Box \bigwedge \Delta[\overline{a}]$, then there is a $c \in D_w^{\mathcal{M}}$ such that $\mathcal{M}, w \vDash \Box\varphi[\overline{a}c]$ for all $\varphi \in \Gamma$, where x is assigned c.

$\Diamond$-type If for each finite subset Δ of $\Gamma(\overline{x})$ such that $|\overline{x}| = |\overline{a}|$, $\mathcal{M}, w \vDash \Diamond \bigwedge \Delta[\overline{a}]$, then there is a v such that wRv and $\mathcal{M}, v \vDash \varphi[\overline{a}]$ for each $\varphi \in \Gamma$.

Theorem

For $\exists\Box$-saturated models $\mathcal{M}, \mathcal{N}$ and $|\overline{a}| = |\overline{b}|$:
$\mathcal{M}, w\overline{a} \leftrightarroweq_{\exists\Box} \mathcal{N}, v\overline{b} \Leftrightarrow \mathcal{M}, w\overline{a} \equiv_{\mathsf{MLMS}^{\approx}} \mathcal{N}, v\overline{b}$

Theorem (Wang TARK17)

A first-order modal formulas is equivalent to a formula in the $\exists\Box$-fragment iff it is invariant under $\exists\Box$-bisimulation.

A complete epistemic logic over S5 models SMLMS

Over S5 (constant-domain) models, **MLMS** is very powerful, it can also express *mention-all* by $\forall x\Diamond(\Box\varphi \vee \Box\neg\varphi)$ (also $\forall x\Box\varphi$ by $\forall x\Diamond\Box\varphi$).

Axioms

TAUT	all axioms of propositional logic
DISTK	$\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$
T	$\Box\varphi \to \varphi$
4MS	$\exists x\Box\varphi \to \Box\exists x\Box\varphi$
5MS	$\neg\exists x\Box\varphi \to \Box\neg\exists x\Box\varphi$
KtoMS	$\Box(\varphi[y/x]) \to \exists x\Box\varphi$ (admissible $\varphi[y/x]$)
MStoK	$\exists x\Box\varphi \to \Box\varphi$ (if $x \notin FV(\varphi)$)
MStoMSK	$\exists x\Box\varphi \to \exists x\Box\Box\varphi$
KT	$\Box\top$

Rules:

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
MONOMS	$\dfrac{\vdash \varphi \to \psi}{\vdash \exists x\Box\varphi \to \exists x\Box\psi}$

To treat the equality (if we introduce it), we also need $\texttt{ID} : x \approx x$ and $\texttt{SUBID} : x \approx y \to (\varphi \to \psi)$. We can derive $\texttt{KEQ} : x \approx y \to \Box(x \approx y)$ and $\texttt{KNEQ} : x \not\approx y \to \Box(x \not\approx y)$.

We can axiomatize the logic over arbitrary models without T, 4MS, 5MS, MStoMSK.

Compare with the know-how logic

TAUT	all axioms of propositional logic	MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
DISTK	$\mathcal{K}p \land \mathcal{K}(p \to q) \to \mathcal{K}q$	NECK	$\dfrac{\varphi}{\mathcal{K}\varphi}$
T	$\mathcal{K}p \to p$	EQREPKh	$\dfrac{\varphi \to \psi}{\mathcal{K}h\varphi \to \mathcal{K}h\psi}$
4	$\mathcal{K}p \to \mathcal{K}\mathcal{K}p$	SUB	$\dfrac{\varphi(p)}{\varphi[\psi/p]}$
5	$\neg\mathcal{K}p \to \mathcal{K}\neg\mathcal{K}p$		
AxKtoKh	$\mathcal{K}p \to \mathcal{K}hp$		
AxKhtoKhK	$\mathcal{K}hp \to \mathcal{K}h\mathcal{K}p$		
AxKhtoKKh	$\mathcal{K}hp \to \mathcal{K}\mathcal{K}hp$		
AxKhKh	$\mathcal{K}h\mathcal{K}hp \to \mathcal{K}hp$		
AxKhbot	$\neg\mathcal{K}h\bot$		

Completeness proof

Definition

A set of $\mathbf{MLMS}^{+}$ formulas has $\exists$-property if for each $\exists x \Box \varphi \in \mathbf{MLMS}^{+}$ it contains a "witness" formula $\exists x \Box \varphi \to \Box \varphi[y/x]$ for some $y \in \mathrm{X}^{+}$ where $\varphi[y/x]$ is admissible.

Definition (Canonical model)

The canonical model is a tuple $\langle W^c, D^c, \sim^c, \rho^c \rangle$ where:

- W^c is the set of maximal SMLMS^{+}-consistent sets with $\exists$-property,
- $D^c = \mathrm{X}^{+}$,
- $s \sim^c t$ iff $\Box(s) \subseteq t$ where $\Box(s) := \{\varphi \mid \Box\varphi \in s\}$,
- $\overline{x} \in \rho^c(P, s)$ iff $P\overline{x} \in s$.

It is routine to show that $\sim^c$ is an equivalence relation

Completeness proof

Lemma

If $\Box\psi \notin s \in W^c$ then there exists a $t \in W^c$ such that $s \sim^c t$ and $\neg\psi \in t$.

The witnesses for $\exists\Box$ formulas can be added by using:

$$\vdash_{\mathsf{SMLMS}} (\exists x \Box\varphi \to \Box\psi) \to \Box(\exists x \Box\varphi \to \Box\psi).$$

Lemma

Let σ^* be the assignment such that $\sigma^*(x) = x$ for all $x \in X^+$. For any $\varphi \in$ **MLMS**$^+$*, any $s \in W^c$:*

$$\mathcal{M}^c, s, \sigma^* \vDash \varphi \Leftrightarrow \varphi \in s$$

Each SMLMS consistent set can be extended to an SMLMS$^+$ consistent set.

Axiomatizations over other classes of frames [Xun Wang 21]

Axioms:

TAUT	all axioms of propositional logic
DISTK	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\Box$to$\exists\Box$	$\Box\varphi[y/x] \to \exists x\Box\varphi$ (if $\varphi[y/x]$ is admissible)

Rules:

MP $\dfrac{\varphi, \varphi \to \psi}{\psi}$ NEC $\dfrac{\varphi}{\Box\varphi}$ $R^{l}\Box$to$\exists\Box$ $\dfrac{\Box\varphi \to \psi}{\exists x\Box\varphi \to \psi}(x \notin FV(\psi))$

Plus the corresponding axioms for frame conditions:

$$\mathsf{D} \quad \Box\varphi \to \Diamond\varphi, \mathsf{T} \quad \Box\varphi \to \varphi$$

$$4 \quad \Box\varphi \to \Box\Box\varphi$$

$$5 \quad \neg\Box\varphi \to \Box\neg\Box\varphi$$

What about decidability? (See course by Rybakov & Shkatov)

The situation for first-order modal logic looks hopeless. Simply putting a decidable fragment of first-order logic plus a modality does not work at all. Following results hold also under usual frame conditions.

Language	Decidability	Ref
P^1	undecidable	[Kripke 62]
x, y, p, P^1	undecidable	[Gabbay 93]
x, y, single P^1	undecidable	[Rybakov & Shkatov 19]

The decidable fragments are rare (only one x in $\Box$). Most of the propositional know-wh logics are in the one variable fragment.

Language	Decidability	Ref
single x	decidable	[Segerberg 73]
$x, y / P^1 /$ GF, $\Box_i(x)$	decidable	[Wolter & Zakharyaschev 01]

What about our bundled fragments?

Without any restrictions on the number of variables, the arity of predicates and the number of variables in $\Box$, we have:

$\exists\Box$-fragment of FOML is not only decidable on arbitrary increasing or constant domain models, but also its complexity is *PSPACE*–complete, as the basic propositional modal logic!

The trick: restricting the power of $\forall$ as it can only occur as $\forall x \Diamond$.

It is similar to how modal logic restrict the $\forall$ into a guarded one: $\forall y(xRy \rightarrow \varphi)$

Tableaux (can be viewed as a satisfiability game)

We start from negated normal form (and require some "cleanness"):

$$\varphi ::= P\overline{x} \mid \neg P\overline{x} \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \exists x \Box \varphi \mid \forall x \Diamond \varphi$$

$$\frac{w : \varphi_1 \vee \varphi_2, \Gamma, \sigma}{w : \varphi_1, \Gamma, \sigma \mid w : \varphi_2, \Gamma, \sigma} \ (\vee) \qquad \frac{w : \varphi_1 \wedge \varphi_2, \Gamma, \sigma}{w : \varphi_1, \varphi_2, \Gamma, \sigma} \ (\wedge)$$

Given $n \geq 0, m \geq 1$:

$$\frac{w : \exists x_1 \Box \varphi_1, \ldots, \exists x_n \Box \varphi_n, \forall y_1 \Diamond \psi_1, \ldots, \forall y_m \Diamond \psi_m, l_1 \ldots l_k, \sigma}{\{(wv^{y}_{y_i} : \{\varphi_j \mid 1 \leq j \leq n\}, \psi_i[y/y_i], \sigma') \mid y \in Dom(\sigma'), i \in [1, m]\}} \ (\texttt{BR})$$

Given $n \geq 1, k \geq 0$:

$$\frac{w : \exists x_1 \Box \varphi_1, \ldots, \exists x_n \Box \varphi_n, l_1 \ldots l_k, \sigma}{w : l_1 \ldots l_k, \sigma} \ (\texttt{END})$$

where $\sigma' = \sigma \cup \{(x_j, x_j) \mid j \in [1, n]\}$ and $l_k \in lit$ (the literals).

An example

$$w : \{\exists x \Box (Px \vee Qx) \wedge \forall y \Diamond \neg Qy \wedge \neg Pz, \{(z,z)\} \qquad (\wedge) \times 2$$

$$w : \{\exists x \Box (Px \vee Qx), \forall y \Diamond \neg Qy, \neg Pz\}\{(z,z)\} \qquad (\mathtt{BR})$$

$$wv_y^x : \{Px \vee Qx, \neg Qx\}, \{(x,x),(z,z)\} \qquad (\vee) \qquad wv_y^z : \{Px \vee Qx, \neg Qz\}, \{(x,x),(z,z)\}$$

$$wv_y^x : \{Px, \neg Qx\}, \{(x,x),(z,z)\} \qquad\qquad wv_y^z : \{Qx, \neg Qz\}, \{(x,x),(z,z)\}$$

Theorem (Wang TARK17)

A formula φ in the $\exists\Box$ fragment is satisfiable is satisfiable iff its NNF has an open tableau.

Theorem (Wang TARK17)

A formula φ in the $\exists\Box$ fragment is satisfiable over arbitrary increasing domain models then it has an finite tree model whose depth is linearly bound by the length of φ.

Corollary (Wang TARK17)

Satisfiability checking of $\exists\Box$ fragment over arbitrary increasing domain is PSPACE-complete.

The $\exists\Box$ fragment behaves like the basic propositional modal logic but much more powerful.

Moreover, we can show that:

Theorem (Padmanabha, Ramanujam, Wang FSTTCS18)

The $\exists\Box$-fragment is decidable over arbitrary constant domain models.

Actually we can show that:

Theorem (Padmanabha, Ramanujam, Wang FSTTCS18)

The $\exists\Box$-fragment cannot distinguish increasing domain and constant domain models. The logic is exactly the same over constant domain models or increasing domain models.

Some bad news

The meaning of the world is the separation of wish and fact.

— Gödel

- $\exists\Box$ fragment is undecidable over S5 models: replacing each quantifier in a first-order formula in the prenex form by $\exists x\Box$ or $\forall x\Diamond\Box$ respectively qua satisfiability
- $\forall\Box$ fragment with two unary predicates is undecidable over constant domain models: use $\Diamond(P(x) \wedge Q(y))$ to encode the binary predicate, and use $\forall z_1\Box\, \forall z_2\Box\, (\Diamond^n\Diamond\, (P(z_1) \wedge Q(z_2)) \to \Box^n\Diamond(P(z_1) \wedge Q(z_2)))$ to force uniformity of evaluation.

It is not as robust as propositional modal logic: we are at the edge of first-order expressivity.

However, it give us a new general approach to find many decidable fragments which are expressive.

General picture: full bundled language

Definition

Given a countable set of predicates $\mathcal{P}$ and a countable set of variables X, the bundled fragment of FOML is

$$\varphi ::= P(x_1, \ldots, x_n) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x \Box \varphi \mid \forall x \Box \varphi \mid \Box \exists x \varphi \mid \Box \forall x \varphi$$

Notation: `A`, `E`, and `B` stand for $\forall$, $\exists$, and $\Box$.

We can define all kinds of fragments:

`AB` (forAll-Box): only $\forall x \Box \varphi$. Similarly, `BA`, `EB`, `BE`, etc

`EBBA`: $\exists x \Box \varphi$ and $\Box \forall x \varphi$. Similarly, `ABBA`, `EBBABE`, etc.

Undecidability over increasing domain

Over Increasing domain models, we consider reduction from tiling problem over $\mathbb{N} \times \mathbb{N}$.

The following sentences are crucial:

- $\forall x \exists y \Box$ [x has a horizontal/vertical successor y];
- $\forall x(\Box)\forall y(\Box)\forall z(\Box)$ [“diagonal property”].

`EBBA`, `ABEBBE` can express such formulas.

Theorem

The SAT problems for `EBBA` and `ABEBBE` over increasing domain models are undecidable.

Liu, Padmanabha, Ramanujam, Wang [Info.& Compt. 23]

Over increasing domain models:

Domain	$\forall\Box$	$\exists\Box$	$\Box\forall$	$\Box\exists$	Upper/ Lower Bound
Increasing	✓	✗	✗	✗	PSPACE-complete
	✗	✓	✗	✗	
	✗	✗	✓	✗	
	✗	✗	✗	✓	EXPSPACE/ PSPACE
	✓	✓	✗	✗	EXPSPACE/NEXPTIME
	✗	✗	✓	✓	
	⋆	✓	✓	⋆	Undecidable
	✗	✓	✗	✓	No FMP
	✓	✓	✗	✓	Undecidable
	✓	✗	✓	✓	EXPSPACE/ NEXPTIME
	loosely bundled				

We can also allow $\exists x \beta$ where β is a boolean combination of atomic formulas and modal formulas. Moreover, we can allow a quantifier alternation of the form $\exists x_1 \cdots \exists x_n \ \forall y_1 \cdots \forall y_m \ \beta$.

The fact that the existential quantifiers are outside the scope of universal quantifiers can help us to obtain decidability results over increasing domain models.

Definition (LBF syntax)

The loosely bundled fragment of FOML is the set of all formulas constructed by the following syntax of φ:

$$\begin{aligned} \varphi ::= &\quad \psi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x_1 \dots \exists x_k \forall y_1 \dots \forall y_l \ \psi \\ \psi ::= &\quad P(z_1, \dots z_n) \mid \neg P(z_1, \dots z_n) \mid \psi \wedge \psi \mid \psi \vee \psi \mid \Box\varphi \mid \Diamond\varphi \end{aligned}$$

where $k, l, n \geq 0$ and $P \in \mathrm{Ps}$ has arity n and $x_1, \dots x_k$, $y_1, \dots y_l$, $z_1, \dots, z_n \in X$.

ABBABE Fragment

ABBABE cannot express $\forall x \exists y \Box \alpha$, but $\forall x \exists y \Diamond \alpha$ is allowed.

- It means that the different witnesses y for each x can work on *different* successors.
- The fragment cannot enforce the interaction between x and y at all successors.
- This property can be used to prove that we can reuse the witnesses by creating new successor subtrees as required.
- If $\forall x \exists y \Diamond \varphi$ is satisfiable, then $\exists y_1 \cdots \exists y_n \forall x (\bigvee \Diamond \varphi[y/y_i])(i \in [1, n])$ is satisfiable (where n is bounded).

Liu, Padmanabha, Ramanujam, Wang [Info.& Compt. 23]

Over constant domain models:

Domain	$\forall\Box$	$\exists\Box$	$\Box\forall$	$\Box\exists$	Upper/ Lower Bound
Constant	✓	$\star$	$\star$	$\star$	Undecidable
	$\star$	$\star$	✓	$\star$	
	✗	✓	✗	✗	PSPACE-complete
	✗	✗	✗	✓	No FMP
	✗	✓	✗	✓	

$\exists\Box$ is still the champion over constant domain models!

In these two tables, the decidability of "no FMP" cases are mostly open, except perhaps `EBBE` which is suggested to be decidable (Padmanabha and a student)

Further directions:

- The cases lacking finite model properties.
- What about adding $\approx$ and constant symbols (for decidability)?
- Which frame conditions can be added while keeping the decidability.
- Vary domain models?

Axiomatizations and model theory of various bundled fragments: see Xun Wang and Yuanzhe Yang's work.