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Inquisitive logic as an epistemic logic of knowing how

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In this paper, we present an alternative interpretation of propositional inquisitive logic as an epistemic logic of *knowing how*. In our setting, an inquisitive logic formula α being supported by a state is formalized as *knowing how to resolve* α (more colloquially, *knowing how* α *is true*) holds on the S5 epistemic model corresponding to the state. Based on this epistemic interpretation, we use a dynamic epistemic logic with both know-how and know-that operators to capture the epistemic information behind the innocent-looking connectives in inquisitive logic. We show that the set of valid know-how formulas corresponds precisely to the inquisitive logic. The main result is a complete axiomatization with intuitive axioms using the full dynamic epistemic language. Moreover, we show that the know-how operator and the dynamic operator can both be eliminated without changing the expressivity over models, which is consistent with the modal translation of inquisitive logic existing in the literature. We hope our framework can give an intuitive alternative interpretation to various concepts and technical results in inquisitive logic, and also provide a powerful and flexible tool to handle both the inquisitive reasoning and declarative reasoning in an epistemic context.

1. Introduction

Inquisitive logic captures the valid reasoning patterns of statements and questions in a neat uniform framework [11]. There are two major views on inquisitive logic: one may view it as a non-classical logic as presented in the early days of the field (cf. e.g., [11]); alternatively, as endorsed by various recent works, one can also view the framework as a conservative extension of classical logic taking the inquisitive disjunction and other machinery as new additions on top of the classical ones (cf. e.g., [7,8]). According to the first *non-classical view*, the basic system **InqB** of propositional inquisitive logic is a *weak intermediate logic* that includes all the axioms of intuitionistic logic, but is *not* closed under uniform substitution. Moreover, inquisitive logic has some surprising close connections to some other logics such as Medvedev Logic [11], and

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it can be viewed as a disguised *propositional intuitionistic dependence logic* [14,40]. The second *extension-view* gives flexibility in designing logical systems combining the power of both classical reasoning and inquisitive reasoning. As a rapidly growing field of logic, besides being intensively studied itself, inquisitive logic has been extended widely with various modalities and other non-classical connectives (cf. e.g., [10,12, 19,29]).

The characteristic semantic feature of inquisitive logic is that it is based on the *support* relation over *information states* (or simply *states*), instead of the usual satisfaction relation over *possible worlds* widely used in various classical and non-classical logics. As a logic, **InqB** collects all the valid propositional formulas supported by all the states. Intuitively, to a modal logician, a state in (propositional) inquisitive semantics is simply *a set* of possible worlds. To an epistemic logician, a state can be further viewed as an epistemic (S5) Kripke model, where the agent is not sure which possible world is the actual one.[1] It is very natural to ask whether there is an intrinsic connection between epistemic logic and inquisitive logic, given both are defined over similar models. In fact, there is a syntactic translation from inquisitive logic to modal and epistemic logic (e.g., [7, Section 5.4]). In this paper, we focus on whether we can have an intuitive epistemic interpretation semantically.

Our approach is based on the non-classical view of inquisitive logic. Thanks to the intimate connection between inquisitive logic and Medvedev logic that we mentioned, we can make use of a crucial observation about the epistemic interpretation of intuitionistic and intermediate logics. Inspired by the original finite-problem semantics of Medvedev logic [22], Wang [37] proposed to interpret intuitionistic truth of a formula α as *knowing how to prove/solve* α. Similar informal ideas of understanding intuitionistic truth as knowledge-how appeared a few times in the literature of intuitionistic logic in the past century (cf. e.g., [23]), starting from the very first paper by Heyting explaining the intuitive meaning of intuitionistic statements [1]:

> To satisfy the intuitionistic demands, the statement must be the realization of the expectation expressed by the proposition p. Here, then, is the *Brouwerian statement* of p: It is known how to prove p.

In contrast with the previous informal philosophical discussions, we now have a formal way to capture this interpretation of intuitionistic truth not merely conceptually but also mathematically, based on the techniques for epistemic logics of *know-wh* proposed and studied by Wang (cf. e.g., [34,36]). The main idea is to introduce the so-called *bundled modalities*, which pack a quantifier and an epistemic modality together to formalize the *de re* knowledge expressed by *knowing how/why/what and so on* (cf. e.g., [35,39]). For example, *knowing how to prove* α (written as $\mathsf{Kh}\alpha$) can be rendered as *there exists* a proof ρ such that *it is known that* ρ proves α. Bundled modalities also lead to bundled fragments of first-order modal logic, which are often decidable [21,27,34].

Combining this technique with some formalized BHK-interpretation of intuitionistic logic may allow us to turn intuitionistic logic and various intermediate logics into epistemic logics of *knowing how*. The general method is to use a powerful epistemic language based on classical logic to *unload* the implicit epistemic content hidden behind the propositional language of propositional intuitionistic logic, foreseen by Hintikka and van Benthem viewing intuitionistic logic as an *implicit* epistemic logic [18,30]. An intuitionistic logic formula α is first translated into a know-how formula $\mathsf{Kh}\alpha$ in our setting; then, depending on the structure of α and the BHK-interpretation, we can further "decode" α by reducing its complexity within the logical language, e.g., $\mathsf{Kh}(\alpha \vee \beta)$ can be decomposed into $(\mathsf{Kh}\alpha \vee \mathsf{Kh}\beta)$, where the connectives outside the scope of Kh are *classical*. This also helps us to understand the distinct role of the negation as the bridge between the classical and the intuitionistic settings. Such an epistemic approach can make intuitionistic logic and its relatives more intuitive, and the existing important technical results become more transparent.

[1] It is interesting to note that *information states* in game theory are precisely those epistemically indistinguishable possible worlds.

In this paper, we apply such ideas to propositional inquisitive logic as a variant of intuitionistic logic under the non-classical view mentioned above. Note that the *intended* interpretation of inquisitive logic does not make reference to knowledge, thus what we are proposing is an *alternative* epistemic interpretation. Instead of talking about *proofs* and *solutions* in intuitionistic logic, here we are concerned with *resolutions* of issues raised by inquisitive formulas. Roughly speaking, a formula can be true in various ways, e.g., $\alpha \vee \beta$ can be true because of the truth of α or the truth of β. We say it is *resolved* if it is settled with a particular way of being true. In a nutshell, we interpret *"s supports* α*"* in inquisitive semantics as *"knowing how to resolve* α*"* (or simply *"knowing how* α *is true"*) over the epistemic model corresponding to s. For example, *"s supports* $\alpha \vee \beta$*"* becomes *"knowing how* $\alpha \vee \beta$ *is true"* over the corresponding epistemic model, which intuitively requires either *knowing how* α *is true* or *knowing how* β *is true*, thus *resolving* the question raised by $\alpha \vee \beta$. This interpretation is also extended to the entailment relation, i.e., α entails β in inquisitive logic is interpreted as *knowing how* α *is true* entails *knowing how* β *is true*, thus interpreting the reasoning in inquisitive logic as *know-how preserving.* This can be considered as the analog of correspondence of entailment in propositional logic and the standard epistemic logic of knowing that.

Actually, the idea of interpreting inquisitive formulas in terms of *knowing how* first appeared in Ciardelli's master thesis in the early days of inquisitive logic [5]. In [11], the authors also made it more precise by using a notion of *realization* (or, say, *resolution*) inspired by the BHK-interpretation of intuitionistic logic, to which we will come back in Section 6 with detailed discussions.

Technically, compared to intuitionistic logic, there is a crucial simplification in inquisitive logic that each atomic proposition p has one and only one possible resolution p. This is due to the assumption in inquisitive semantics that atomic propositions stand for statements, and questions are formed only via question-forming operators like the inquisitive disjunction (cf. [7, Section 2.5.5]). Translated into our epistemic interpretation, this assumes that you always *know how* to resolve an atomic formula if you *know that* it is true. On the other hand, *knowing that* p *is true* is a presupposition of *knowing how it is true.* Therefore, *knowing how* to resolve the atomic proposition p is equivalent to *knowing that* p is *resolvable* in our epistemic rendering of inquisitive logic. Note that this cannot be extended to the case of an arbitrary formula α, which would trivialize the know-how formulas. Nevertheless, as we will see later, this simplification leads to the characteristic features (and the charm) of inquisitive logic. Based on this assumption about atomic propositions, we will eventually be able to eliminate the *knowing how* operator in our "epistemicization" of the inquisitive logic. All these will become more clear, after our technical framework is established.

Here we summarize our main contributions in this paper:

- From an epistemic point of view, we propose a dynamic-epistemic logical framework of *knowing how* to give an alternative interpretation of inquisitive logic.
- We show that the propositional inquisitive logic, as a weak intermediate logic, is exactly the valid know-how fragment of our logic.
- We obtain a complete axiomatization of the full dynamic epistemic logic with intuitive axioms, which can transparently explain the axioms and results of inquisitive logic from our epistemic perspective.
- We show that both the knowing how modality and the dynamic modality can be eliminated in terms of expressivity, and thus inquisitive logic can be viewed as a fragment of epistemic logic.

More conceptually, with our approach, we want to bridge:

- ***possible worlds and states*** by viewing states as epistemic models to base our semantics on.
- ***classical and non-classical logics*** by using a modal logic based on classical connectives to interpret the non-classical inquisitive logic.

- ***non-modal and modal formulations*** by using the epistemic language of knowing how to reveal the rich dynamic-epistemic information behind the propositional formulas of inquisitive logic.
- ***de re and de dicto knowledge*** by connecting the *knowledge-how* with *knowledge-that* relevant to inquisitive reasoning.

We hope this epistemic approach can further enhance the power of inquisitive logic to wider applications while keeping its spirit. This may make inquisitive logic more intuitive to an audience who are not familiar with non-classical logics and support-based semantics. Future directions are discussed in Section 7.

Structure of the paper. We recall the language and semantics of propositional inquisitive logic in Section 2. Section 3 introduces the language and semantics of our dynamic epistemic logic of knowing how, and shows that it captures the original inquisitive logic precisely as a fragment. In Section 4, we obtain an intuitive axiomatization for the full language and show its completeness by using the reduction axioms. Section 5 looks at the core concepts in inquisitive semantics from our intuitive epistemic perspective. In Section 6, we discuss the related work, particularly about inquisitive modal logic. We conclude in Section 7 with future directions.

2. Preliminaries: inquisitive logic

We first present some basic definitions and results of (propositional) inquisitive logic **InqB** following the expositions of [7,11].

Definition 1 *(Language* **PL***).* Given a countable set $\mathbf{P}$ of proposition letters, the language of propositional logic ($\mathbf{PL}^{\mathbf{P}}$) is defined as follows:

$$\alpha ::= p \mid \bot \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha) \mid (\alpha \to \alpha)$$

where $p \in \mathbf{P}$. For abbreviations, we write $\neg\alpha$ for $\alpha \to \bot$, $\top$ for $\neg\bot$, and $\alpha \leftrightarrow \beta$ for $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

$\mathbf{P}$ is assumed to be countable as in [7]. We write $\mathbf{PL}$ for $\mathbf{PL}^{\mathbf{P}}$ when $\mathbf{P}$ is given in the context. For readability, in the rest of the paper, we often omit the parentheses if no ambiguity arises.

Instead of the satisfaction relation based on possible worlds, **InqB** adopts the following *support relation* in its semantics based on states, which makes **InqB** behave non-classically.

While the *states* were initially defined via the concept of *index* [4], it was later defined as a subset of the world set of a *(propositional) information model* in the recent literature (cf. e.g., [7]), here we adopt the latter definition.[2]

Definition 2 *(Information model).* Given $\mathbf{P}$, an (information) model is a pair $\mathcal{M} = \langle W, V\rangle$ where:

- W is a non-empty set of possible worlds;
- $V : W \to \wp(\mathbf{P})$ is a valuation function.

Given $\mathcal{M}$, we refer to its components as $W_{\mathcal{M}}$ and $V_{\mathcal{M}}$. A *full model* is a model such that $V[W] = \wp(\mathbf{P})$.

Following [11], the inquisitive semantics of $\mathbf{PL}$ over information models is given by the *support* relation.

[2] Note that in contrast with [7], we do not allow W to be empty in order to ease the later presentation without changing any technical result.

Definition 3 *(Support).* Given **P** and an information model $\mathcal{M} = \langle W, V \rangle$, an *(information) state* $s \subseteq W$ is a subset of W. W and $\varnothing$ are called *trivial state* and *inconsistent state* respectively. *Support* is a relation between states and formulas (written as $\mathcal{M}, s \Vdash \alpha$):

1. $\mathcal{M}, s \Vdash p$ iff $\forall w \in s, p \in V(w)$.
2. $\mathcal{M}, s \Vdash \bot$ iff $s = \varnothing$.
3. $\mathcal{M}, s \Vdash (\alpha \wedge \beta)$ iff $\mathcal{M}, s \Vdash \alpha$ and $\mathcal{M}, s \Vdash \beta$.
4. $\mathcal{M}, s \Vdash (\alpha \vee \beta)$ iff $\mathcal{M}, s \Vdash \alpha$ or $\mathcal{M}, s \Vdash \beta$.
5. $\mathcal{M}, s \Vdash (\alpha \to \beta)$ iff $\forall t \subseteq s$: if $\mathcal{M}, t \Vdash \alpha$ then $\mathcal{M}, t \Vdash \beta$.

We write $s \Vdash \alpha$ for $\mathcal{M}, s \Vdash \alpha$ when no confusion arises.

Definition 4 *(Entailment ($\Vdash$)).* A set of **PL**-formulas Γ entails a **PL**-formula α in inquisitive semantics, $\Gamma \Vdash \alpha$, if and only if for any state s in any model $\mathcal{M}$ if $\mathcal{M}, s \Vdash \Gamma$ then $\mathcal{M}, s \Vdash \alpha$. We say α is *valid* if $\Vdash \alpha$.

Definition 5 *(Inquisitive logic).* Inquisitive logic, **InqB**, is the set of **PL**-formulas that are valid in inquisitive semantics, i.e., the set of formulas that are supported by all states in all models.

It is straightforward to show:

Proposition 6 *([11]). For any* **PL***-formula α, $\alpha \in$* **InqB** *iff for any full model $\mathcal{M}$, $\mathcal{M}, W_{\mathcal{M}} \Vdash \alpha$.*

Now we present the proof system SINTDN and two axioms schemata $\mathtt{KP}$ and $\mathtt{ND}_k$, which in combination give rise to two distinct axiomatizations of **InqB**.

System SINTDN

Axioms		**Rules:**	
INTU	Intuitionistic validities	MP	$\dfrac{\alpha, \alpha \to \beta}{\beta}$
DNp	$\neg\neg p \to p$ for all $p \in \mathbf{P}$		

Let $\mathtt{KP}$ and $\mathtt{ND}_k$ be the following axiom schemata:

$$\mathtt{KP} \quad (\neg\alpha \to \beta \vee \gamma) \to (\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma)$$
$$\mathtt{ND}_k \quad (\neg\alpha \to \textstyle\bigvee_{1 \le i \le k} \neg\beta_i) \to \textstyle\bigvee_{1 \le i \le k}(\neg\alpha \to \neg\beta_i)$$

Theorem 7 *(Axiomatizations of* **InqB** *[11]).* $\mathsf{SINTDN} + \mathtt{KP}$ *and* $\mathsf{SINTDN} + \{\mathtt{ND}_k \mid k \in \mathbb{N}\}$ *are both sound and complete for* **InqB**.

The following definitions are taken from [11] adapted with a given model $\mathcal{M}$ as in [7].[3] These characterize some important concepts in inquisitive semantics. Note that in the earlier literature (cf. [11]), the definition of proposition was the set of alternatives. But the latest definition [9] has been modified as the downward closure of the old one. We adopt the latest definition.

Definition 8 *(Alternatives and propositions).* Let α be a **PL** formula and let $\mathcal{M}$ be a model.

- An *alternative* for α in $\mathcal{M}$ is a maximal state s in $\mathcal{M}$ supporting α;

[3] Following the new notion in [7], we call *possibilities* (Definition 2.9 in [11]) as *alternatives*, and call *assertions* (Definition 2.14 in [11]) as *statements*. Note that in Definition 2.14 in [11], questions and statements (assertions) are defined absolutely with respect to a full model with the trivial state. Here we generalized it to a relative notion as in the cases of inquisitiveness and informativeness.

- The *proposition* expressed by α in $\mathcal{M}$ (call it $[\alpha]_{\mathcal{M}}$), is the set of states in $\mathcal{M}$ that supports α. That is, $[\alpha]_{\mathcal{M}} = \{s \in W_{\mathcal{M}} \mid s \Vdash \alpha\}$.

Definition 9 *(Inquisitiveness and informativeness).* Let α be a **PL** formula and let $\mathcal{M}$ be a model.

- α is *inquisitive* in $\mathcal{M}$ if $[\alpha]_{\mathcal{M}}$ contains at least two alternatives;
- α is *informative* in $\mathcal{M}$ if there is some world in $\mathcal{M}$ that is not included in any alternative for α in $\mathcal{M}$.

Furthermore, we can define the relative notions of *statements* and *questions*.

Definition 10 *(Questions and statements).* Given a model $\mathcal{M}$:

- α is a *question* in $\mathcal{M}$ iff it is not informative in $\mathcal{M}$;
- α is a *statement* in $\mathcal{M}$ iff it is not inquisitive in $\mathcal{M}$.

We will come back to these in Section 5 from our epistemic perspective. Before that, we shall introduce our framework of the logic of knowing how to interpret inquisitive logic epistemically.

3. Inquisitive logic as a logic of knowing how

In this section, we give a formal epistemic interpretation of inquisitive logic by using an epistemic logic of knowing how.

3.1. Language and models

We first introduce our dynamic epistemic language of knowing how, where on top of **PL** we add three modalities of know-that, know-how, and updates.

Definition 11 *(Language* **DELKh***).* Given a countable set of proposition letters **P**, the *Dynamic Epistemic Language of Knowing How* ($\mathbf{DELKh^P}$) is defined as follows:

$$\varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi) \mid \mathsf{K}\varphi \mid \mathsf{Kh}\alpha \mid \Box\varphi$$

where $p \in \mathbf{P}$ and $\alpha \in \mathbf{PL^P}$. For abbreviations, we write $\widehat{\mathsf{K}}$ for $\neg\mathsf{K}\neg$, and write $\Diamond$ for $\neg\Box\neg$. We denote the Kh-free fragment as $\mathbf{DEL^P}$, and the $\Box$-free fragment of $\mathbf{DEL^P}$ as $\mathbf{EL^P}$.

Again, we often omit the **P** from $\mathbf{DELKh^P}$ when **P** is fixed in the context. Intuitively, $\mathsf{K}\varphi$ expresses that "the agent *knows that* φ", $\mathsf{Kh}\alpha$ says that "the agent *knows how* to resolve α" or simply "the agent *knows how* α is true", and $\Box\varphi$ says that "φ holds, no matter what further information is given". Note that Kh only takes **PL**-formulas α whereas K and $\Box$ can be combined with any **DELKh**-formulas φ. Therefore we can express $\mathsf{K}\neg\mathsf{Kh}\alpha$ but not $\mathsf{Kh}\mathsf{K}\alpha$ in **DELKh**.

DELKh will be interpreted on standard *single-agent epistemic models* where the (implicit) epistemic relation is the *total relation*. Technically speaking, **such models are exactly the information models** as we defined in Definition 2, when we omit the epistemic relation since it is always total.[4] For this reason, in the rest of the paper, we will also call information models *epistemic models*.

[4] Nevertheless, in a multi-agent setting, explicit epistemic relations are necessary. We leave it for a future occasion.

For **notational convenience**, we also write $w \in \mathcal{M}$ in case that $w \in W_{\mathcal{M}}$, $\mathcal{M}' \subseteq \mathcal{M}$ in case that $\mathcal{M}'$ is a submodel of $\mathcal{M}$. If $w \in \mathcal{M}' \subseteq \mathcal{M}$ then we write $(\mathcal{M}', w) \subseteq (\mathcal{M}, w)$.

Although the models are simply $\mathsf{S5}$ epistemic models, in order to reflect the non-classical features of **InqB**, we define the semantics for Kh via the resolutions, based on the idea that knowing how α is true means knowing a particular resolution for α, in line with the BHK-interpretation of the intuitionistic connectives. We first define the resolution space for each formula below in Definition 12. The actual resolutions on each world for each formula, to be defined in Definition 14, will be subsets of the resolution space.

Definition 12 *(Resolution space).* S is a function assigning each α its set of potential resolutions:

$$\begin{aligned} S(p) &= \{p\}, \text{ for } p \in \mathbf{P} \\ S(\bot) &= \{\bot\} \\ S(\alpha \vee \beta) &= (S(\alpha) \times \{0\}) \cup (S(\beta) \times \{1\}) \\ S(\alpha \wedge \beta) &= S(\alpha) \times S(\beta) \\ S(\alpha \to \beta) &= S(\beta)^{S(\alpha)} \end{aligned}$$

Let $\mathcal{S} = \bigcup_{\alpha \in \mathbf{PL}} S(\alpha)$.

Our definition is similar to the definition of realizations in [25, Part 4], which was not proposed in the context of inquisitive logic. The definition of resolution space reflects the intuition that for each α, there is a set of possible ways to *resolve it as an issue* or say to *make it true*. Intuitively, the set of potential resolutions of a disjunction is the disjoint union of the potential resolutions of each disjunct: to make a disjunction true you need to explicitly make one of the disjuncts true. The resolution space of a conjunction is the Cartesian product of the resolution space of each conjunct: to make a conjunction true, you need to make both conjuncts true. The resolution space for an implication, is the set of functions from the resolution space of the antecedent to the resolution space of the consequent: to make an implication true, you need to have a way to transform any resolution of the antecedent to some resolution of the consequent.

Note that the potential resolutions for atomic propositions are assumed to be singletons, which reflects the underlying assumption in inquisitive semantics that atomic propositions do not bring inquisitiveness themselves. In other words, we always *know how* to resolve atomic propositions when possible. This is *the* most fundamental difference between **InqB** and other intermediate logics such as Medvedev logic. It will become more clear when we discuss the axioms later. Technically speaking, the possible resolution of each atomic proposition p is not necessary p itself, as long as it is unique, and it will become more clear when the semantics is introduced.

For technical convenience, following the definition of resolution space for atomic propositions, the resolution space of $\bot$ is defined as $\{\bot\}$, but as we will see later $\bot$ does not have any real resolution.

It is obvious that the resolution space for each $\alpha \in \mathbf{PL}$ is non-empty and finite:

Proposition 13. *For any $\alpha \in \mathbf{PL}$, $S(\alpha) \neq \varnothing$ and it is finite.*

Now given a model, we can generate the *(actual) resolutions* of each α on each world.

Definition 14 *(Resolution).* Given a model $\mathcal{M}$, the resolution function $R : W \times \mathbf{PL} \to \mathcal{S}$ is defined as follows:

$$R(w, \bot) = \varnothing$$

$$R(w, p) = \begin{cases} \{p\} & \text{if } p \in V_{\mathcal{M}}(w) \\ \varnothing & \text{otherwise} \end{cases}$$

$$R(w, \alpha \vee \beta) = (R(w, \alpha) \times \{0\}) \cup (R(w, \beta) \times \{1\})$$
$$R(w, \alpha \wedge \beta) = R(w, \alpha) \times R(w, \beta)$$
$$R(w, \alpha \to \beta) = \{f \in S(\beta)^{S(\alpha)} \mid f[R(w, \alpha)] \subseteq R(w, \beta)\}$$

For $U \subseteq W$, we write $R(U, \alpha)$ for $\bigcap_{w \in U} R(w, \alpha)$. When $U = W_{\mathcal{M}}$ we also write $R(\mathcal{M}, \alpha)$ for $R(U, \alpha)$.

We call $R(w, \alpha)$ the set of resolutions for an issue α on a given world w. It is clear that $R(w, \alpha) \subseteq S(\alpha)$.

Note that although the resolution space of $\bot$ is non-empty for technical convenience, it cannot have any actual resolution on any world. An atomic proposition p can have the resolution p only when it is true on w. The resolutions for an implication $\alpha \to \beta$ on a possible world are the functions in the resolution space of $\alpha \to \beta$ mapping a resolution of α to a resolution of β on the same world in line with the BHK-interpretation.

Recall that $\neg\alpha$ is the abbreviation of $\alpha \to \bot$. Since negation plays an important role in intermediate logics, we have the following observation, which is useful for later discussions.

Proposition 15. *For any $\mathcal{M}, w$, any α, $R(w, \neg\alpha)$ is either $\varnothing$ or a fixed singleton set independent from w, and $R(w, \neg\alpha) = \varnothing$ iff $R(w, \alpha) \neq \varnothing$.*

Proof. By the definitions of R and S:

$$\begin{aligned}
&R(w, \alpha \to \bot) \\
=&\{f \in S(\bot)^{S(\alpha)} \mid f[R(w, \alpha)] \subseteq R(w, \bot)\} \\
=&\{f \in \{\bot\}^{S(\alpha)} \mid f[R(w, \alpha)] \subseteq \varnothing\} \\
=&\{f \in \{\bot\}^{S(\alpha)} \mid f[R(w, \alpha)] = \varnothing\} \\
=&\begin{cases} \varnothing & \text{if } R(w, \alpha) \neq \varnothing \\ \{f_\bot^\alpha\} & \text{if } R(w, \alpha) = \varnothing \end{cases}
\end{aligned}$$

where $f_\bot^\alpha$ is the constant function such that $f_\bot^\alpha(x) = \bot$ for any $x \in S(\alpha)$. Note that $f_\bot^\alpha$ only depends on α and it is independent from w. ■

It is also important that $R(w, \alpha)$ only depends on the valuation on w itself but not on other worlds.

Proposition 16. *For any $\mathcal{M}, w$ and $\mathcal{N}, v$, if $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$ then $R(w, \alpha) = R(v, \alpha)$ for all $\alpha \in$ **PL**.*

3.2. Semantics

Given the definition of resolutions, we can define the satisfaction relation of **DELKh** on possible worlds in epistemic models. Note that the connectives outside the scope of Kh are classical and the semantics of K is standard as in epistemic logic. The semantics of $\mathsf{Kh}\alpha$ on a world w is the formalization of the idea that the agent knows how to resolve α iff it knows a particular resolution of α. The semantics of the dynamic modality $\Box$ is based on taking submodels as in a version of arbitrary announcement logic [2], representing informational updates in terms of eliminating possibilities.

Definition 17 *(Semantics).* For any $\varphi \in$ **DELKh** and a pointed model $\mathcal{M}, w$ where $\mathcal{M} = \langle W, V \rangle$, the satisfaction relation is defined as below:

$\mathcal{M}, w \nvDash \bot$		
$\mathcal{M}, w \vDash p$	$\iff$	$p \in V(w)$
$\mathcal{M}, w \vDash (\varphi \vee \psi)$	$\iff$	$\mathcal{M}, w \vDash \varphi$ or $\mathcal{M}, w \vDash \psi$
$\mathcal{M}, w \vDash (\varphi \wedge \psi)$	$\iff$	$\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$
$\mathcal{M}, w \vDash (\varphi \to \psi)$	$\iff$	$\mathcal{M}, w \vDash \varphi$ implies $\mathcal{M}, w \vDash \psi$
$\mathcal{M}, w \vDash \Box\varphi$	$\iff$	for any $(\mathcal{M}', w) \subseteq (\mathcal{M}, w), \mathcal{M}', w \vDash \varphi$
$\mathcal{M}, w \vDash \mathsf{K}\varphi$	$\iff$	for any $v \in \mathcal{M}, \mathcal{M}, v \vDash \varphi$
$\mathcal{M}, w \vDash \mathsf{Kh}\alpha$	$\iff$	there exists an $x \in S(\alpha)$ s.t. for any $v \in \mathcal{M}, x \in R(v, \alpha)$

We say a formula is *valid on* $\mathcal{M}$ ($\mathcal{M} \vDash \varphi$) if $\mathcal{M}, w \vDash \varphi$ for all $w \in \mathcal{M}$. We say a set of **DELKh**-formula Γ *entails* another formula φ ($\Gamma \vDash \varphi$), if for any pointed model $\mathcal{M}, w$, $\mathcal{M}, w \vDash \Gamma$ implies $\mathcal{M}, w \vDash \varphi$. We say φ is *valid* ($\vDash \varphi$) if $\varnothing \vDash \varphi$. A formula schema is valid if all its instances are valid.

It is not hard to see that the structure of the truth condition of Kh is in terms of the bundle $\exists x\mathsf{K}$ as in other know-wh logics [34,36]. Recall that $R(\mathcal{M}, \alpha) = (\bigcap_{v \in \mathcal{M}} R(v, \alpha))$ (cf. Definition 14). The semantics of Kh can be then reformulated as below for notational brevity.

$$\boxed{\mathcal{M}, w \vDash \mathsf{Kh}\alpha \iff R(\mathcal{M}, \alpha) \neq \varnothing}$$

Note that the truth conditions of K and Kh do not depend on the designated world, therefore we have:

Proposition 18. *For any model* $\mathcal{M}, w$*:*

- $\mathcal{M}, w \vDash \mathsf{Kh}\alpha \iff \mathcal{M} \vDash \mathsf{Kh}\alpha$, *and* $\mathcal{M}, w \vDash \neg\mathsf{Kh}\alpha \iff \mathcal{M} \vDash \neg\mathsf{Kh}\alpha$*;*
- $\mathcal{M}, w \vDash \mathsf{K}\alpha \iff \mathcal{M} \vDash \mathsf{K}\alpha$, *and* $\mathcal{M}, w \vDash \neg\mathsf{K}\alpha \iff \mathcal{M} \vDash \neg\mathsf{K}\alpha$.

As a consequence, the introspection axioms $\mathsf{Kh}\alpha \leftrightarrow \mathsf{KKh}\alpha$ *and* $\neg\mathsf{Kh}\alpha \leftrightarrow \mathsf{K}\neg\mathsf{Kh}\alpha$ *are valid.*

It is clear that the above semantics is simply classical for $\alpha \in \mathbf{PL}$. In particular, $\{\alpha \in \mathbf{PL} \mid \vDash \alpha\}$ is classical propositional logic (**CPL**). However, let **InqKhL** be $\{\alpha \mid \vDash \mathsf{Kh}\alpha\} \subseteq \mathbf{PL}$, we will show that **InqKhL** = **InqB**. Before that, to understand the semantics of Kh better, we first show that the classical semantics of α can be viewed from the perspective of resolutions: *resolvability equals truth*. Note that for atomic formulas p, $\mathcal{M}, w \vDash p \iff R(w, p) \neq \varnothing$ is obviously true, however we need to generalize it to any formula $\alpha \in \mathbf{PL}$.

Proposition 19. *For any* $\alpha \in \mathbf{PL}$ *and pointed model* $\mathcal{M}, w$. $\mathcal{M}, w \vDash \alpha \iff R(w, \alpha) \neq \varnothing$, *where* $\alpha \in \mathbf{PL}$.

Proof. Induction on the structure of α:

$$\begin{aligned}
\mathcal{M}, w \vDash p &\iff p \in V_{\mathcal{M}}(w) \iff R(w,p) = \{p\} \iff R(w,p) \neq \varnothing \\
\mathcal{M}, w \nvDash \bot &\iff R(w, \bot) = \varnothing \\
\mathcal{M}, w \vDash (\alpha \vee \beta) &\iff \mathcal{M}, w \vDash \alpha \text{ or } \mathcal{M}, w \vDash \beta \iff R(w,\alpha) \neq \varnothing \text{ or } R(w,\beta) \neq \varnothing \\
&\iff \text{there exists an } x \in R(w,\alpha) \text{ or there exists a } y \in R(w,\beta) \\
&\iff \text{there exists a pair } \langle x, 0\rangle \text{ or } \langle y, 1\rangle \text{ in } R(w, \alpha \vee \beta) \\
&\iff R(w, \alpha \vee \beta) \neq \varnothing \\
\mathcal{M}, w \vDash (\alpha \wedge \beta) &\iff \mathcal{M}, w \vDash \alpha \text{ and } \mathcal{M}, w \vDash \beta \iff R(w,\alpha) \neq \varnothing \text{ and } R(w,\beta) \neq \varnothing \\
&\iff \text{there exists an } x \in R(w,\alpha) \text{ and there exists a } y \in R(w,\beta)
\end{aligned}$$

$$\begin{aligned}
&\iff \text{ there exists a pair } \langle x,y\rangle \in R(w,\alpha\wedge\beta)\\
&\iff R(w,\alpha\wedge\beta)\neq\varnothing\\
\mathcal{M},w\vDash(\alpha\to\beta) &\iff \mathcal{M},w\vDash\alpha \text{ implies } \mathcal{M},w\vDash\beta \iff R(w,\alpha)\neq\varnothing \text{ implies } R(w,\beta)\neq\varnothing\\
&\iff \text{ there exists an} f\in S(\beta)^{S(\alpha)} \text{ such that } f[R(w,\alpha)]\subseteq R(w,\beta)\\
&\iff R(w,\alpha\to\beta)\neq\varnothing \quad \blacksquare
\end{aligned}$$

For negations, based on Propositions 19 and 15, we have

$$\mathcal{M},w\vDash\neg\alpha \iff R(w,\alpha)=\varnothing \iff \mathcal{M},w\nvDash\alpha.$$

Now, based on Proposition 19, we have an alternative semantics for $\mathsf{K}\alpha$ for $\alpha\in\mathbf{PL}$:

$$\boxed{\mathcal{M},w\vDash\mathsf{K}\alpha \iff \textbf{for any } v\in\mathcal{M},\ \textbf{there exists} \text{ an } x\in R(v,\alpha)}$$

Compared with the truth condition of Kh, it now becomes clear that the distinction between Kh and K is exactly the distinction between the *de re* and *de dicto* knowledge, i.e., knowing α is resolvable vs. knowing how α is resolved.

$$\boxed{\mathcal{M},w\vDash\mathsf{Kh}\alpha \iff \textbf{there exists} \text{ an } x \text{ s.t. } \textbf{for any } v\in\mathcal{M}, x\in R(v,\alpha)}$$

Based on this distinction, $\mathsf{Kh}\alpha$ is clearly stronger than $\mathsf{K}\alpha$:

Proposition 20. $\mathsf{Kh}\alpha\to\mathsf{K}\alpha$ *is valid for all* $\alpha\in\mathbf{PL}$.

The distinction disappears if we consider the atomic propositions since there can be at most one fixed resolution for each $p\in\mathbf{PL}$.

Proposition 21. $\mathsf{Kh}p\leftrightarrow\mathsf{K}p$ *is valid for all* $p\in\mathbf{P}$.

However, $\mathsf{K}\alpha\to\mathsf{Kh}\alpha$ does not hold in general.

Example 1. $\mathsf{Kh}(p\vee\neg p)\leftrightarrow\mathsf{K}(p\vee\neg p)$ is not valid, e.g., in the model $\langle\{w,v\},V\rangle$ where $p\in V(w)$ but $p\notin V(v)$, $\mathsf{K}(p\vee\neg p)$ holds everywhere but $\mathsf{Kh}(p\vee\neg p)$ holds nowhere.

The example also shows that the valid Kh-formulas are not closed under uniform substitution, to which we will come back in Section 4.

It is well-known that in intermediate logics negation plays a role in bridging the classical and the intuitionitic validities, we show that this can be understood by the fact that $\neg$ bridges Kh and K in our setting.

Proposition 22 *(Negative translation).* $\mathsf{Kh}\neg\alpha\leftrightarrow\mathsf{K}\neg\alpha$ *is valid. As a consequence,* $\mathsf{Kh}\neg\neg\alpha\leftrightarrow\mathsf{K}\alpha$ *is valid.*

Proof. By (the proof of) Proposition 15

$$R(v,\neg\alpha)\neq\varnothing \iff R(v,\neg\alpha)=\{f^{\alpha}_{\perp}\} \iff R(v,\alpha)=\varnothing$$

Thus $\mathcal{M},w\vDash\mathsf{Kh}\neg\alpha \iff R(\mathcal{M},\neg\alpha)\neq\varnothing \iff R(v,\alpha)=\varnothing$ for all $v\in\mathcal{M} \iff \mathcal{M},w\vDash\mathsf{K}\neg\alpha$. Therefore $\mathcal{M},w\vDash\mathsf{Kh}\neg\neg\alpha \iff \mathcal{M},w\vDash\mathsf{K}\neg\neg\alpha \iff \mathcal{M},w\vDash\mathsf{K}\alpha$. $\blacksquare$

Although we cannot reduce Kh to K in general, we can reduce the complexity of α in $\mathsf{Kh}\alpha$ step by step, which will play an important role in the later sections.

Proposition 23. *The following formulas and schemata are valid:*

- $\mathsf{Kh}\bot \leftrightarrow \bot$
- $\mathsf{Kh}(\alpha \vee \beta) \leftrightarrow \mathsf{Kh}\alpha \vee \mathsf{Kh}\beta$
- $\mathsf{Kh}(\alpha \wedge \beta) \leftrightarrow \mathsf{Kh}\alpha \wedge \mathsf{Kh}\beta$
- $\mathsf{Kh}(\alpha \to \beta) \leftrightarrow \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$

Proof. We only show the non-trivial cases of $\mathsf{Kh}(\alpha \vee \beta)$ and $\mathsf{Kh}(\alpha \to \beta)$.

- $\mathcal{M}, w \vDash \mathsf{Kh}(\alpha \vee \beta) \iff R(\mathcal{M}, \alpha \vee \beta) \neq \varnothing$
  $\iff$ there exists an $\langle x, 0\rangle \in R(\mathcal{M}, \alpha \vee \beta)$ or there exists a $\langle y, 1\rangle \in R(\mathcal{M}, \alpha \vee \beta)$
  $\iff$ there exists an $x \in R(\mathcal{M}, \alpha)$ or there exists a $y \in R(\mathcal{M}, \beta)$
  $\iff \mathcal{M}, w \vDash \mathsf{Kh}\alpha$ or $\mathcal{M}, w \vDash \mathsf{Kh}\beta \iff \mathcal{M}, w \vDash \mathsf{Kh}\alpha \vee \mathsf{Kh}\beta$
- Let us now consider the case for $\mathsf{Kh}(\alpha \to \beta) \leftrightarrow \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$.
  $\Longrightarrow$: Suppose $\mathcal{M}, w \vDash \mathsf{Kh}(\alpha \to \beta)$, then by the semantics, there is some $f \in R(\mathcal{M}, \alpha \to \beta)$. Towards a contradiction, suppose $\mathcal{M}, w \nvDash \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$. That is, there is an $v \in \mathcal{M}$ and an $\mathcal{M}', v \subseteq \mathcal{M}, v$ s.t. $\mathcal{M}', v \vDash \mathsf{Kh}\alpha$ but $\mathcal{M}', v \nvDash \mathsf{Kh}\beta$. So there is an $x \in R(\mathcal{M}', \alpha)$. Recall that f is a function with domain $S(\alpha)$, and $S(\alpha) \supseteq R(u, \alpha)$ for all $u \in \mathcal{M}'$, thus $x \in Dom(f)$. Moreover, since $f \in R(\mathcal{M}, \alpha \to \beta)$, $f \in R(\mathcal{M}', \alpha \to \beta)$. Let $y = f(x)$. By the definition of $R(\mathcal{M}', \alpha \to \beta)$, $y \in R(u, \beta)$ for each $u \in \mathcal{M}'$. Therefore $\mathcal{M}', v \vDash \mathsf{Kh}\beta$, a contradiction.
  $\Longleftarrow$: Suppose $\mathcal{M}, w \vDash \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$, then for all $v \in \mathcal{M}$, $\mathcal{M}, v \vDash \Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$. By the semantics of $\Box$, for any $v \in \mathcal{M}$ and for any $\mathcal{M}', v \subseteq \mathcal{M}, v$, $\mathcal{M}', v \vDash \mathsf{Kh}\alpha \to \mathsf{Kh}\beta$ $(*)$. By Proposition 13, $S(\alpha)$ is finite and non-empty, thus we can assume $S(\alpha) = \{x_0, x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$. For $i \in \{0, \ldots, n\}$, let $W_i = \{w \mid x_i \in R(w, \alpha)\}$. If W_i is not empty then let $\mathcal{M}_i$ be the submodel of $\mathcal{M}$ such that $W_{\mathcal{M}_i} = W_i$. Clearly $x_i \in R(W_i, \alpha)$, therefore for any $u \in \mathcal{M}_i$, $\mathcal{M}_i, u \vDash \mathsf{Kh}\alpha$. By $(*)$ we have $\mathcal{M}_i, u \vDash \mathsf{Kh}\beta$ thus there is a $y_i \in R(W_i, \beta)$. Now fix a $y \in S(\beta) \neq \varnothing$, let $f = \{\langle x_i, y_i\rangle \mid i \in \{0, \ldots, n\}$ and $W_i \neq \varnothing\} \cup \{\langle x_i, y\rangle \mid i \in \{0, \ldots, n\}$ and $W_i = \varnothing\}$. Clearly $f \in S(\beta)^{S(\alpha)}$. Now for any $v \in \mathcal{M}$ and $i \in \{0, \ldots, n\}$, if $x_i \in R(w, \alpha)$ then $v \in W_i$ by the definition of W_i, thus $y_i \in R(v, \beta)$ by the construction of f. Therefore $f[R(v, \alpha)] \subseteq R(v, \beta)$ for all $v \in \mathcal{M}$. It follows that $\mathcal{M}, v \vDash \mathsf{Kh}(\alpha \to \beta)$ for all $v \in \mathcal{M}$ including w. Note that the axiom of choice is not needed in the above finitary constructions. ∎

Remark 1. Propositions 21 and 23 will help us to eliminate the Kh modalities without changing the expressive power. Actually, we can also eliminate the $\Box$ modality eventually. We will discuss the reduction formally in Section 4 when discussing the axiomatization featuring the corresponding reduction axioms.

Now we have an intuitive reading of $\alpha \vee \neg\alpha$ in inquisitive logic based on Propositions 23 and 22.

$$\mathcal{M}, w \vDash \mathsf{Kh}(\alpha \vee \neg\alpha) \iff \mathcal{M}, w \vDash \mathsf{Kh}\alpha \vee \mathsf{Kh}\neg\alpha \iff \mathcal{M}, w \vDash \mathsf{Kh}\alpha \vee \mathsf{K}\neg\alpha$$

The formula $\mathsf{Kh}\alpha \vee \mathsf{K}\neg\alpha$ says either you know how to resolve α or you know it is not resolvable/true, and it is clearly not valid in general. This explains intuitively why inquisitive logic does not accept the law of excluded middle.

Given the validity of $\mathsf{Kh}(\alpha \to \beta) \leftrightarrow \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$ and Proposition 18, we can give an alternative compositional truth condition to $\mathsf{Kh}(\alpha \to \beta)$, which is more handy to use.

Proposition 24. *For any model $\mathcal{M}$, $\mathcal{M}, w \vDash \mathsf{Kh}(\alpha \to \beta)$ iff for any $\mathcal{M}' \subseteq \mathcal{M}$, $\mathcal{M} \vDash \mathsf{Kh}\alpha$ implies $\mathcal{M} \vDash \mathsf{Kh}\beta$.*

Proof. It suffices to show that $\mathcal{M}, w \vDash \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$ iff for any $\mathcal{M}' \subseteq \mathcal{M}$, $\mathcal{M} \vDash \mathsf{Kh}\alpha$ implies $\mathcal{M} \vDash \mathsf{Kh}\beta$, which follows directly from the semantics of $\mathsf{K}\Box$. Note that Given a pointed model $\mathcal{M}, w$, $\Box$ quantifies over all the submodels of $\mathcal{M}'$ such that $w \in \mathcal{M}'$. Since on an $S5$ model, K refers to all the points $w \in \mathcal{M}$, the combination of $\mathsf{K}\Box$ quantifies over all the submodels of $\mathcal{M}$. That is, for any $\mathcal{M}' \subseteq \mathcal{M}$ and $v \in \mathcal{M}'$, $\mathcal{M}', v \vDash \mathsf{Kh}\alpha$ implies $\mathcal{M}', v \vDash \mathsf{Kh}\beta$. By Proposition 18 the proof is completed. ■

In the following, given $\Gamma \subseteq \mathbf{PL}$, let $\mathsf{Kh}\Gamma = \{\mathsf{Kh}\gamma \mid \gamma \in \Gamma\}$. As another consequence of the validity of $\mathsf{Kh}(\alpha \to \beta) \leftrightarrow \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$, we have:

Proposition 25. $\mathsf{Kh}\Gamma \vDash \mathsf{Kh}(\alpha \to \beta)$ *iff* $\mathsf{Kh}\Gamma \vDash \mathsf{Kh}\alpha \to \mathsf{Kh}\beta$. *As a special case,* $\vDash \mathsf{Kh}(\alpha \to \beta)$ *iff* $\vDash \mathsf{Kh}\alpha \to \mathsf{Kh}\beta$.

Proof. It suffices to show that $\mathsf{Kh}\Gamma \vDash \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$ iff $\mathsf{Kh}\Gamma \vDash \mathsf{Kh}\alpha \to \mathsf{Kh}\beta$ $\Longrightarrow$ is based on the fact that $\mathsf{K}\varphi \to \varphi$ and $\Box\varphi \to \varphi$ are valid. $\Longleftarrow$ is based on the fact that if $\mathcal{M}, w \vDash \mathsf{Kh}\Gamma$ then any for submodel $\mathcal{M}'$ of $\mathcal{M}$, $\mathcal{M}' \vDash \mathsf{Kh}\Gamma$ by the semantics of Kh. Indeed, if for all the worlds in $\mathcal{M}$, there exists a uniform resolution for each formula in Γ, the same resolutions will certainly serve as the uniform resolutions for worlds in $\mathcal{M}'$. Therefore, supposing $\mathsf{Kh}\Gamma \vDash \mathsf{Kh}\alpha \to \mathsf{Kh}\beta$ and $\mathcal{M}, w \vDash \mathsf{Kh}\Gamma$, it follows that $\mathsf{Kh}\alpha \to \mathsf{Kh}\beta$ is satisfied on all the submodels of $\mathcal{M}$. By Proposition 18, that is, for any $\mathcal{M}' \subseteq \mathcal{M}$, $\mathcal{M} \vDash \mathsf{Kh}\alpha$ implies $\mathcal{M} \vDash \mathsf{Kh}\beta$. By Proposition 24, we have $\mathcal{M}, w \vDash \mathsf{Kh}(\alpha \to \beta)$. ■

Based on Proposition 25, we have the following theorem for Kh formulas, which is the counterpart of Proposition 3.10 (deduction theorem) in [11].

Theorem 26. *For any* $\alpha \in \mathbf{PL}$ *and* $\Gamma, \Gamma' \subseteq \mathbf{PL}$ *such that* Γ *is finite,*

$$\mathsf{Kh}(\Gamma' \cup \Gamma) \vDash \mathsf{Kh}\alpha \iff \mathsf{Kh}\Gamma' \vDash \mathsf{Kh}(\bigwedge_{\gamma \in \Gamma} \gamma \to \alpha).$$

Proof.

$$\begin{aligned}
\mathsf{Kh}(\Gamma' \cup \Gamma) \vDash \mathsf{Kh}\alpha &\iff \text{for any } \mathcal{M}, w \text{ such that } \mathcal{M}, w \vDash \mathsf{Kh}\Gamma' \cup \mathsf{Kh}\Gamma, \text{ then } \mathcal{M}, w \vDash \mathsf{Kh}\alpha \\
&\iff \text{for any } \mathcal{M}, w \text{ such that } \mathcal{M}, w \vDash \mathsf{Kh}\Gamma', \text{ if } \mathcal{M}, w \vDash \bigwedge_{\gamma \in \Gamma} \mathsf{Kh}\gamma, \text{ then } \mathcal{M}, w \vDash \mathsf{Kh}\alpha \\
&\iff \text{for any } \mathcal{M}, w \text{ such that } \mathcal{M}, w \vDash \mathsf{Kh}\Gamma', \mathcal{M}, w \vDash \bigwedge_{\gamma \in \Gamma} \mathsf{Kh}\gamma \to \mathsf{Kh}\alpha \\
&\iff \mathsf{Kh}\Gamma' \vDash \bigwedge_{\gamma \in \Gamma} \mathsf{Kh}\gamma \to \mathsf{Kh}\alpha \\
&\iff \mathsf{Kh}\Gamma' \vDash \mathsf{Kh} \bigwedge_{\gamma \in \Gamma} \gamma \to \mathsf{Kh}\alpha \qquad \text{(by Proposition 23)} \\
&\iff \mathsf{Kh}\Gamma' \vDash \mathsf{Kh}(\bigwedge_{\gamma \in \Gamma} \gamma \to \alpha) \qquad \text{(by Proposition 25)} \quad \blacksquare
\end{aligned}$$

As an analog of the *persistence* of inquisitive formulas over sub-states [11, Prop. 2.4], we show the persistence of $\mathsf{Kh}\alpha$ in our setting with $\Box$. Intuitively, once we know how to resolve α, we will not forget, even given more information.

Proposition 27 *(Persistence).* $\mathsf{Kh}\alpha \leftrightarrow \Box\mathsf{Kh}\alpha$ *is valid for any* $\alpha \in \mathbf{PL}$.

Proof. $\Longrightarrow$: Suppose $\mathcal{M}, w \vDash \mathsf{Kh}\alpha$ then $R(\mathcal{M}, \alpha) \neq \varnothing$. It is clear that $R(\mathcal{M}, \alpha) \subseteq R(\mathcal{M}', \alpha)$ for any submodel $\mathcal{M}'$. Therefore $R(\mathcal{M}', \alpha) \neq \varnothing$ for any submodel $\mathcal{M}'$, thus $\mathcal{M}, w \vDash \Box\mathsf{Kh}\alpha$.

$\Longleftarrow$: It is trivial since a model is also a submodel of itself. ■

Another important interaction property between Kh and $\Box$ is the following, which can be compared to Proposition 2.5 in [11] regarding the singleton state.

Proposition 28. *$\alpha \leftrightarrow \Diamond\mathsf{Kh}\alpha$ is valid for any $\alpha \in \mathbf{PL}$.*

Proof. $\Longrightarrow$: It is valid because we can always go to a singleton submodel containing the current world only, where $\mathsf{K}\alpha$ holds. Note that for singleton models, i.e., models with only one world, $\mathsf{K}\alpha \to \mathsf{Kh}\alpha$ holds trivially for any α since any actual resolution on that single world will be the uniform resolution.

$\Longleftarrow$ is based on the fact that the updates do not change the truth values of propositional formulas on the current world, and the validity of $\mathsf{Kh}\alpha \to \mathsf{K}\alpha$ and $\mathsf{K}\alpha \to \alpha$. ■

As a feature distinguishing intuitionistic logic and classical logic, disjunction property is also an important property of inquisitive logic [11, Prop. 3.9]. It holds naturally in our logic.

Proposition 29 *(Disjunction property). For any formulas $\alpha, \beta \in \mathbf{PL}$, $\vDash \mathsf{Kh}(\alpha \vee \beta) \iff \vDash \mathsf{Kh}\alpha$ or $\vDash \mathsf{Kh}\beta$.*

Proof. $\Longrightarrow$: Suppose $\nvDash \mathsf{Kh}\alpha$ and $\nvDash \mathsf{Kh}\beta$, then we have some models $\mathcal{M}, w$ and $\mathcal{N}, v$ such that $\mathcal{M}, w \nvDash \mathsf{Kh}\alpha$ and $\mathcal{N}, v \nvDash \mathsf{Kh}\beta$. Now we can simply merge the two models together as the disjoint union $\mathcal{M} \uplus \mathcal{N}$. By the semantics of Kh it is clear that $\mathcal{M} \uplus \mathcal{N}, w \nvDash \mathsf{Kh}\alpha \vee \mathsf{Kh}\beta$. By Proposition 23, $\nvDash \mathsf{Kh}(\alpha \vee \beta)$.

$\Longleftarrow$ is trivial by Proposition 23. ■

As some examples, we show the validity of Kh-versions of some axioms and valid formulas in **InqB** (cf. Theorem 7).

Proposition 30. *The following are valid:*

`KhDNp`	$\mathsf{Kh}(\neg\neg p \to p)$ *for* $p \in \mathbf{P}$
`KhPEIRCEp`	$\mathsf{Kh}(((p \to q) \to p) \to p)$ *for* $p, q \in \mathbf{P}$
`KhKP`	$\mathsf{Kh}((\neg\alpha \to \beta \vee \gamma) \to (\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma))$
`KhND`$_k$	$\mathsf{Kh}((\neg\alpha \to \bigvee_{1 \le i \le k} \neg\beta_i) \to \bigvee_{1 \le i \le k}(\neg\alpha \to \neg\beta_i))$

Proof. For `KhDNp`: by Proposition 25, it suffices to check $\vDash \mathsf{Kh}\neg\neg p \to \mathsf{Kh}p$. By Proposition 22, it amounts to check $\vDash \mathsf{K}p \to \mathsf{Kh}p$, which is valid by Proposition 21.

For `KhPEIRCEp`: by Proposition 25, it suffices to check $\vDash \mathsf{Kh}((p \to q) \to p) \to \mathsf{Kh}p$ which amounts to $\vDash \mathsf{K}\Box(\mathsf{K}\Box(\mathsf{Kh}p \to \mathsf{Kh}q) \to \mathsf{Kh}p) \to \mathsf{Kh}p$. By Proposition 21, we just need to check $\vDash \mathsf{K}\Box(\mathsf{K}\Box(\mathsf{K}p \to \mathsf{K}q) \to \mathsf{K}p) \to \mathsf{K}p$. Now suppose $\mathcal{M}, w \nvDash \mathsf{K}p$ then there is $v \in \mathcal{M}$ such that $\mathcal{M}, v \vDash \neg p$. We need to show that $\mathcal{M}, w \nvDash \mathsf{K}\Box(\mathsf{K}\Box(\mathsf{K}p \to \mathsf{K}q) \to \mathsf{K}p)$. Take the singleton submodel $\mathcal{M}'$ of $\mathcal{M}$ with the world v only, then it is clear that $\mathcal{M}', v \nvDash \mathsf{K}\Box(\mathsf{K}p \to \mathsf{K}q) \to \mathsf{K}p$. Therefore $\mathcal{M}, w \nvDash \mathsf{K}\Box(\mathsf{K}\Box(\mathsf{K}p \to \mathsf{K}q) \to \mathsf{K}p)$.

For `KhKP`: By Proposition 25, we only need to check $\vDash \mathsf{Kh}(\neg\alpha \to \beta \vee \gamma) \to \mathsf{Kh}((\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma))$. It amounts to $\vDash \mathsf{K}\Box(\mathsf{K}\neg\alpha \to (\mathsf{Kh}\beta \vee \mathsf{Kh}\gamma)) \to \mathsf{K}\Box(\mathsf{K}\neg\alpha \to \mathsf{Kh}\beta) \vee \mathsf{K}\Box(\mathsf{K}\neg\alpha \to \mathsf{Kh}\gamma)$ based on Proposition 23. We prove its contraposition. Suppose $\mathcal{M}, w \vDash \neg\mathsf{K}\Box(\mathsf{K}\neg\alpha \to \mathsf{Kh}\beta) \wedge \neg\mathsf{K}\Box(\mathsf{K}\neg\alpha \to \mathsf{Kh}\gamma)$ then there are two submodels $\mathcal{M}'_1$ and $\mathcal{M}'_2$ such that $\mathcal{M}'_1 \vDash \mathsf{K}\neg\alpha \wedge \neg\mathsf{Kh}\beta$ and $\mathcal{M}'_2 \vDash \mathsf{K}\neg\alpha \wedge \neg\mathsf{Kh}\gamma$. Then the union $\mathcal{M}'_1 \cup \mathcal{M}'_2$ is a submodel of $\mathcal{M}$ making $\mathsf{K}\neg\alpha \to (\mathsf{Kh}\beta \vee \mathsf{Kh}\gamma)$ false. Thus $\mathcal{M}, w \nvDash \mathsf{K}\Box(\mathsf{K}\neg\alpha \to (\mathsf{Kh}\beta \vee \mathsf{Kh}\gamma))$.

The validity of `KhND`$_k$ can be proved similarly as in the case of `KhKP`. ■

Note that Peirce's schema $\mathsf{Kh}(((\alpha \to \beta) \to \alpha) \to \alpha)$ is not valid in general. For example, take the instance where $\alpha = p \vee \neg p$ and $\beta = p$ and the full model is a counterexample.

So far, we have seen the αs in the valid $\mathsf{Kh}\alpha$ formulas behave pretty much like the valid formulas in the inquisitive logic, we will show it is no coincidence.

3.3. **InqKhL** = **InqB**

In this subsection, we show $\mathbf{InqKhL} = \{\alpha \in \mathbf{PL} \mid \vDash \mathsf{Kh}\alpha\}$ is exactly the inquisitive logic **InqB**. We will actually prove a stronger result showing the corresponding semantics consequences are the same.

Definition 31. Given any model $\mathcal{M} = \langle W, V\rangle$ and a non-empty state $s \subseteq W$, let $\mathcal{M}_s$ be the submodel $\langle s, V|_s\rangle$ where $V|_s$ is the restriction of V on s, that is, $V(w) = V|_s(w)$ for any $w \in s$.

Here is an easy observation that the worlds outside the given state are irrelevant according to support semantics. This also justifies our notion of $s \Vdash \alpha$ without specifying the model $\mathcal{M}$ below.

Proposition 32. *Given any two models $\mathcal{M}$ and $\mathcal{M}'$, and a state $s \subseteq W_{\mathcal{M}}$ and $s \subseteq W_{\mathcal{M}'}$, if $\mathcal{M}_s = \mathcal{M}'_s$, then for any $\alpha \in$ **PL**: $\mathcal{M}, s \Vdash \alpha \iff \mathcal{M}', s \Vdash \alpha$. In particular if s is non-empty and $s \subseteq W_{\mathcal{M}}$, then $\mathcal{M}, s \Vdash \alpha \iff \mathcal{M}_s, s \Vdash \alpha$.*

Now we establish the logical equivalence between $\mathcal{M}$ and $(\mathcal{M}, W_{\mathcal{M}})$ where $W_{\mathcal{M}}$ is viewed as the trivial state.

Lemma 33. *Given any $\alpha \in$ **PL** and any pointed model $\mathcal{M}, w$,*

$$\mathcal{M}, w \vDash \mathsf{Kh}\alpha \iff \mathcal{M} \vDash \mathsf{Kh}\alpha \iff \mathcal{M}, W_{\mathcal{M}} \Vdash \alpha.$$

Proof. By Proposition 18, we only need to show that for any model $\mathcal{M}$, $\mathcal{M} \vDash \mathsf{Kh}\alpha$ iff $W_{\mathcal{M}} \Vdash \alpha$. We prove the lemma by induction on α.

$$\begin{aligned}
\mathcal{M} \vDash \mathsf{Kh}p &\iff \mathcal{M} \vDash \mathsf{K}p \iff \text{ for each } w \in W_{\mathcal{M}}, p \in V(w) \iff W_{\mathcal{M}} \Vdash p.\\
\mathcal{M} \nvDash \mathsf{Kh}\bot &\iff W_{\mathcal{M}} \neq \varnothing \iff W_{\mathcal{M}} \nVdash \bot.\\
\mathcal{M} \vDash \mathsf{Kh}(\alpha \vee \beta) &\iff \mathcal{M} \vDash \mathsf{Kh}\alpha \vee \mathsf{Kh}\beta \iff \mathcal{M} \vDash \mathsf{Kh}\alpha \text{ or }\\
\mathcal{M} \vDash \mathsf{Kh}\beta &\iff W_{\mathcal{M}} \Vdash \alpha \text{ or } W_{\mathcal{M}} \Vdash \beta \iff W_{\mathcal{M}} \Vdash \alpha \vee \beta.\\
\mathcal{M} \vDash \mathsf{Kh}(\alpha \wedge \beta) &\iff \mathcal{M} \vDash \mathsf{Kh}\alpha \wedge \mathsf{Kh}\beta \iff \mathcal{M} \vDash \mathsf{Kh}\alpha \text{ and }\\
\mathcal{M} \vDash \mathsf{Kh}\beta &\iff W_{\mathcal{M}} \Vdash \alpha \text{ and } W_{\mathcal{M}} \Vdash \beta \iff W_{\mathcal{M}} \Vdash \alpha \wedge \beta.\\
\mathcal{M} \vDash \mathsf{Kh}(\alpha \to \beta) &\iff \text{ for any } \mathcal{M}' \subseteq \mathcal{M}, \mathcal{M}' \vDash \mathsf{Kh}\alpha \text{ implies } \mathcal{M}' \vDash \mathsf{Kh}\beta \text{ (by Proposition 24)}\\
&\iff \text{ for any non-empty } s \subseteq W_{\mathcal{M}}, s \Vdash \alpha \text{ implies } s \Vdash \beta \text{ (by IH)}\\
&\iff W_{\mathcal{M}} \Vdash \alpha \to \beta. \quad \blacksquare
\end{aligned}$$

Since $s = W_{(\mathcal{M}_s)}$ for any non-empty state s, as a consequence of Lemma 33 and Proposition 32, we have:

Proposition 34. *For each non-empty state s in $\mathcal{M}$, any $\alpha \in$ **PL**, $\mathcal{M}, s \Vdash \alpha \iff \mathcal{M}_s \vDash \mathsf{Kh}\alpha$.*

Now we are ready to establish the relation between **InqB** and **InqKhL**.

Theorem 35. *Given any $\{\alpha\} \cup \Gamma \subseteq$ **PL**, $\Gamma \Vdash \alpha$ iff $\mathsf{Kh}\Gamma \vDash \mathsf{Kh}\alpha$.*

Proof. First, note that due to Proposition 18, $\mathsf{Kh}\Gamma \vDash \mathsf{Kh}\alpha$ iff for any $\mathcal{M}$, $\mathcal{M} \vDash \mathsf{Kh}\Gamma$ implies $\mathcal{M} \vDash \mathsf{Kh}\alpha$.

$\Longrightarrow$: For any model $\mathcal{M}$ s.t. $\mathcal{M} \vDash \mathsf{Kh}\Gamma$, we have $W_{\mathcal{M}} \Vdash \Gamma$ by Lemma 33. Since $\Gamma \Vdash \alpha$, it follows that $W_{\mathcal{M}} \Vdash \alpha$. By Lemma 33 again, we have $\mathcal{M} \vDash \mathsf{Kh}\alpha$. Therefore $\mathsf{Kh}\Gamma \vDash \mathsf{Kh}\varphi$.

$\Longleftarrow$: For any model $\mathcal{M}$ and state $s \subseteq W_{\mathcal{M}}$ s.t. $s \Vdash \Gamma$, we need to show $s \Vdash \alpha$. If $s = \varnothing$, since the empty state supports all formulas in inquisitive semantics, it follows that $s \Vdash \alpha$. If s is non-empty, by Proposition 34, $\mathcal{M}_s \vDash \mathsf{Kh}\Gamma$, thus $\mathcal{M}_s \vDash \mathsf{Kh}\alpha$. By Proposition 34 again, $s \Vdash \alpha$. As a result, $\Gamma \Vdash \alpha$. ∎

When $\Gamma = \varnothing$, it follows immediately that:

Corollary 36. **InqB** = **InqKhL**.

4. Axiomatizing the full logic

We showed in the previous section that **InqKhL** = **InqB**, thus the valid Kh-fragment can be axiomatized by the corresponding axioms for inquisitive logic. However, the more interesting question to answer is what the logic with respect to the full language **DELKh** is. In this section, we provide a complete axiomatization and also show that the full **DELKh**-language is equally expressive as the epistemic fragment with K modality only. The conceptual advantage of our axiomatization is that all the axioms are epistemically intuitive, compared to the axioms of inquisitive logic. The axiomatization also shows the hidden dynamic-epistemic content of inquisitive logic in a clear syntactic manner.

System SDELKh

Axioms

TAUT	Propositional tautologies	KhK	$\mathsf{Kh}\alpha \to \mathsf{K}\alpha$
$\mathtt{DIST}_{\mathsf{K}}$	$\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$	KKhp	$\mathsf{K}p \to \mathsf{Kh}p$
$\mathtt{T}_{\mathsf{K}}$	$\mathsf{K}\varphi \to \varphi$	Kh⊥	$\mathsf{Kh}\bot \leftrightarrow \bot$
4_{K}	$\mathsf{K}\varphi \to \mathsf{KK}\varphi$	Kh∨	$\mathsf{Kh}(\alpha \vee \beta) \leftrightarrow \mathsf{Kh}\alpha \vee \mathsf{Kh}\beta$
5_{K}	$\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\varphi$	Kh∧	$\mathsf{Kh}(\alpha \wedge \beta) \leftrightarrow \mathsf{Kh}\alpha \wedge \mathsf{Kh}\beta$
$\mathtt{DIST}_{\Box}$	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	Kh→	$\mathsf{Kh}(\alpha \to \beta) \leftrightarrow \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$
$\mathtt{T}_{\Box}$	$\Box\varphi \to \varphi$	4_{Kh}	$\mathsf{Kh}\alpha \to \mathsf{KKh}\alpha$
$4_{\Box}$	$\Box\varphi \to \Box\Box\varphi$	5_{Kh}	$\neg\mathsf{Kh}\alpha \to \mathsf{K}\neg\mathsf{Kh}\alpha$
PR	$\mathsf{K}\Box\varphi \to \Box\mathsf{K}\varphi$	$\mathtt{EU}_k$	$\alpha \wedge \bigwedge_{1\le i\le k} \widehat{\mathsf{K}}(\alpha \wedge \alpha_i) \to \Diamond(\mathsf{K}\alpha \wedge \bigwedge_{1\le i\le k} \widehat{\mathsf{K}}\alpha_i)$
Per	$\alpha \to \Box\alpha$		$(k \in \mathbb{N},\ \alpha_i \in \mathbf{PL}$ for $i \in \mathbb{N})$
Ver	$\alpha \to \Diamond\mathsf{Kh}\alpha$	where	$p \in \mathbf{P}$, $\alpha, \beta \in \mathbf{PL}$, $\varphi \in \mathbf{DELKh}$

Rules:

MP $\dfrac{\varphi, \varphi \to \psi}{\psi}$ $\quad \mathtt{NEC}_{\mathsf{K}}$ $\dfrac{\vdash \varphi}{\vdash \mathsf{K}\varphi}$ $\quad \mathtt{NEC}_{\Box}$ $\dfrac{\vdash \varphi}{\vdash \Box\varphi}$

S5 axiom schemata/rules for K and S4 axiom schemata/rules for $\Box$ are expectable. PR is the axioms of *perfect recall* often assumed in temporal epistemic logic and dynamic epistemic logic (cf. e.g., [38]). Per says the truth values of propositional formulas do not change given informational updates. Ver says that propositional truth is eventually verifiable. For any finite set of worlds in the current model, $\{\mathtt{EU}_k \mid k \in \mathbb{N}\}$ ensures the existence of an updated submodel that contains exactly the current world and that set of worlds. We will use $\mathtt{EU}_k$ to prove the reduction formula BK∨ in Appendix A. KhK says that know-how is stronger than know-that. KKhp, Kh⊥, Kh∨, Kh∧, Kh→ are the reduction axioms decoding the inquisitive formulas. Introspection schemata 4_{K}, 4_{Kh} and 5_{Kh} can be proved from the rest of the system. In particular, 4_{Kh} requires an inductive proof on the structure of α. We include them for the sake of their intuitive meanings.

Theorem 37 *(Soundness).* SDELKh *is sound over the class of all epistemic models.*

Proof. The validity of S5 axiom schemata/rules for K and S4 axiom schemata/rules for $\Box$ are immediate. Based on Propositions 18, 20, 21, 28, and 23, we only need to check PR and Per. For PR, it is easier to verify its dual form $\Diamond\widehat{\mathsf{K}}\varphi \to \widehat{\mathsf{K}}\Diamond\varphi$: if there is a submodel where φ holds at some world then there is a world and a submodel including it where φ holds. Per is valid since moving to any submodel does not change the valuation of the current world. $\mathtt{EU}_k$ is valid because as long as the current world sees a (finite) set of worlds, we can take the submodel that contains both the current world and the set of worlds as the witness for $\Diamond$. ■

Note that we do not have the rule of uniform substitution for this system in general (recall Example 1). Moreover, even the rule of monotonicity for Kh is not valid, e.g., $\vDash \neg\neg\alpha \to \alpha$ but $\nvDash \mathsf{Kh}\neg\neg\alpha \to \mathsf{Kh}\alpha$. However, since we have TAUT and the modalities of $\Box$ and K are *normal*, we can have the admissible rule rRE of *replacement of equals by equals* if we treat all the $\mathsf{Kh}\alpha$ as atomic formulas when doing the substitutions:

$$\texttt{rRE}: \quad \frac{\vdash \varphi \leftrightarrow \psi}{\vdash \chi[\varphi/\psi] \leftrightarrow \chi} \quad \text{given that the substitution does not happen in the scope of Kh.}$$

Due to $\mathtt{NEC}_\mathsf{K}$, $\mathtt{NEC}_\Box$ and the axiom Kh$\to$, another useful admissible rule is the syntactic analog of (the non-trivial side of) Proposition 25:

$$\texttt{RKh}\!\to: \quad \frac{\vdash \mathsf{Kh}\alpha \to \mathsf{Kh}\beta}{\vdash \mathsf{Kh}(\alpha \to \beta)}$$

To prove the completeness we need some extra provable (technical) formulas inspired by the reduction of the arbitrary announcement operator in [2] in the single-agent case.

Proposition 38. *The following schemata are provable in* SDELKh, *where* $\alpha \in \mathbf{PL}$ *and* $\varphi \in \mathbf{DEL}$ *(i.e.,* Kh*-free).*

INV	$\Box\alpha \leftrightarrow \alpha$	B$\vee$	$\Box(\alpha \vee \varphi) \leftrightarrow \alpha \vee \Box\varphi$
KINV	$\Box\mathsf{K}\alpha \leftrightarrow \mathsf{K}\alpha$	BK$\vee$	$\Box(\widehat{\mathsf{K}}\alpha \vee \mathsf{K}\alpha_1 \vee \cdots \vee \mathsf{K}\alpha_n) \leftrightarrow \alpha \vee \mathsf{K}(\alpha \vee \alpha_1) \vee \cdots \vee \mathsf{K}(\alpha \vee \alpha_n)$
hKINV	$\Box\widehat{\mathsf{K}}\alpha \leftrightarrow \alpha$		

Proof. INV is the combination of Per and $\mathtt{T}_\Box$. KINV is proved from Per and PR. hKINV is a special instance of BK$\vee$ when $n = 0$. We include the (tedious) proofs of the other two formulas in Appendix A. ■

Recall that **DEL** is the Kh-free fragment of **DELKh**, and **EL** is the $\Box$-free fragment of **DEL**. By the following lemmata, we show that each **DELKh**-formula is provably equivalent to an **EL**-formula.

Lemma 39. *Each* **DELKh***-formula is provably equivalent to a* **DEL** *formula in* SDELKh.

Proof. Note that with the help of rRE, we can repeatedly apply Axioms Kh$\bot$, Kh$\wedge$, Kh$\vee$, Kh$\to$ step by step to reduce $\mathsf{Kh}\alpha$ to simpler Kh formulas. It is not hard to show that eventually, all the Kh-formula can be reduced to some formulas with $\mathsf{Kh}p$ only. By Axioms KhK and KKhp, we have $\vdash \mathsf{Kh}p \leftrightarrow \mathsf{K}p$, which will eventually eliminate any Kh modality completely. ■

Lemma 40. *Each* **DEL***-formula is provably equivalent to an* **EL** *formula in* SDELKh.

Proof. Here we follow the idea in [2]. We say a formula $\varphi \in \mathbf{EL}$ is in normal form if it is a conjunction of disjunctions of the form $\alpha \vee \widehat{\mathsf{K}}\alpha_0 \vee \mathsf{K}\alpha_1 \vee \cdots \vee \mathsf{K}\alpha_n$ where $\alpha, \alpha_0, \cdots, \alpha_n \in \mathbf{PL}$. Every formula in single-agent S5 is equivalent to a formula in normal form [24]. So we only need to prove that $\Box(\alpha \vee \widehat{\mathsf{K}}\alpha_0 \vee \mathsf{K}\alpha_1 \vee \cdots \vee \mathsf{K}\alpha_n)$

is provably equivalent to an **EL** formula in SDELKh. Then we can eliminate the $\Box$ step by step from the innermost ones that do not have any $\Box$ in its scope.

By B$\vee$, $\Box(\alpha \vee \widehat{\mathsf{K}}\alpha_0 \vee \mathsf{K}\alpha_1 \vee \cdots \vee \mathsf{K}\alpha_n)$ is equivalent to $\alpha \vee \Box(\widehat{\mathsf{K}}\alpha_0 \vee \mathsf{K}\alpha_1 \vee \cdots \vee \mathsf{K}\alpha_n)$, and by BK$\vee$, $\alpha \vee \Box(\widehat{\mathsf{K}}\alpha_0 \vee \mathsf{K}\alpha_1 \vee \cdots \vee \mathsf{K}\alpha_n)$ is equivalent to $\alpha \vee \alpha_0 \vee \mathsf{K}(\alpha_0 \vee \alpha_1) \vee \cdots \vee \mathsf{K}(\alpha_0 \vee \alpha_n)$, which is a formula in **EL**. Note that B$\vee$ and BK$\vee$ are provable in SDELKh as shown by Proposition 38. ∎

Lemmata 39 and 40 together with Theorem 37 also tell us about the expressivity of **DELKh**.

Theorem 41 *(Expressivity).* **DELKh** *is equally expressive as* **EL** *over epistemic models.*

In particular, we have the following corollary.

Corollary 42. *For each $\alpha \in$ **PL**, there is an epistemic formula $\varphi \in$ **EL** such that*

$$\mathcal{M}, W_{\mathcal{M}} \Vdash \alpha \iff \mathcal{M}, w \vDash \mathsf{Kh}\alpha \iff \mathcal{M}, w \vDash \varphi.$$

As shown in the literature, a simple direct translation from inquisitive logic to epistemic logic will be given in Section 6.

Theorem 43 *(Completeness).* SDELKh *is complete over the class of all epistemic models.*

Proof. We prove the completeness by translating each **DELKh**-formula φ into an equivalent **EL**-formula φ' and follow the strategy below.

$$\vDash \varphi \overset{\text{expressive equivalence}}{\Longrightarrow} \vDash \varphi' \overset{\text{completeness of S5}}{\Longrightarrow} \vdash_{\mathsf{S5}} \varphi' \overset{\mathsf{S5} \subseteq \mathsf{SDELKh}}{\Longrightarrow} \vdash_{\mathsf{SDELKh}} \varphi' \overset{\text{provable equivalence}}{\Longrightarrow} \vdash_{\mathsf{SDELKh}} \varphi \quad \blacksquare$$

Strong completeness can be obtained in the similar process w.r.t. a given assumption set Γ (cf. e.g., [3]). Decidability of SDELKh immediately follows from the proofs of the above theorems and the decidability of S5.

Corollary 44. SDELKh *is decidable.*

To demonstrate the power and the use of our system, we show that the following important axioms can be proved based on our intuitive epistemic axioms. In particular, compared to the semantic validity of KhDNp shown in Proposition 30, the syntactic proof of KhDNp presents the non-trivial use of the axioms regarding $\Box$ and K in Proposition 38.

Proposition 45. *The following are provable in* SDELKh*:*

KKhN	$\mathsf{K}\neg\alpha \to \mathsf{Kh}\neg\alpha$
KhDNp	$\mathsf{Kh}(\neg\neg p \to p)$
KhND$_k$	$\mathsf{Kh}((\neg\alpha \to \bigvee_{1\le i\le k} \neg\beta_i) \to \bigvee_{1\le i\le k}(\neg\alpha \to \neg\beta_i))$

Proof. For KKhN:

$\vdash \mathsf{Kh}\alpha \to \alpha$	KhK, T$_\mathsf{K}$, TAUT	(1)
$\vdash \neg\alpha \to \neg\mathsf{Kh}\alpha$	(1)TAUT	(2)
$\vdash \Box\neg\alpha \to \Box\neg\mathsf{Kh}\alpha$	(2)NEC$_\Box$	(3)
$\vdash \neg\alpha \to \Box\neg\mathsf{Kh}\alpha$	(3)Per, TAUT	(4)

$$\vdash \mathsf{K}\neg\alpha \to \mathsf{K}\Box\neg\mathsf{Kh}\alpha \qquad (4)\mathtt{NEC}_\mathsf{K} \qquad (5)$$
$$\vdash \mathsf{K}\neg\alpha \to \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\bot) \qquad (5)\mathtt{Kh}\bot, \mathtt{rRE} \qquad (6)$$
$$\vdash \mathsf{K}\neg\alpha \to \mathsf{Kh}\neg\alpha \qquad (6)\mathtt{Kh}\to, \mathtt{rRE} \qquad (7)$$

For KhDNp:

$$\vdash \neg\neg p \to p \qquad \mathtt{TAUT} \qquad (1)$$
$$\vdash \mathsf{K}(\neg\neg p \to p) \qquad (1)\mathtt{NEC}_\mathsf{K} \qquad (2)$$
$$\vdash \mathsf{K}\neg\neg p \to \mathsf{K}p \qquad (2)\mathtt{DIST}_\mathsf{K}, \mathtt{MP} \qquad (3)$$
$$\vdash \mathsf{Kh}\neg\neg p \to \mathsf{K}\neg\neg p \qquad \mathtt{KhK} \qquad (4)$$
$$\vdash \mathsf{K}p \to \mathsf{Kh}p \qquad \mathtt{KKhp} \qquad (5)$$
$$\vdash \mathsf{Kh}\neg\neg p \to \mathsf{Kh}p \qquad (4)(3)(5)\mathtt{TAUT} \qquad (6)$$
$$\vdash \mathsf{Kh}(\neg\neg p \to p) \qquad (6)\mathtt{RKh}\to \qquad (7)$$

For $\mathtt{KhND}_k$:

$$\vdash \mathsf{K}\alpha \vee \bigvee_{1\le i\le k} \mathsf{K}(\alpha \vee \neg\beta_i) \to \bigvee_{1\le i\le k} (\mathsf{K}\alpha \vee \mathsf{K}(\alpha \vee \neg\beta_i)) \qquad \mathtt{TAUT} \qquad (1)$$
$$\vdash \mathsf{K}(\alpha \vee \bigvee_{1\le i\le k} \mathsf{K}(\alpha \vee \neg\beta_i)) \leftrightarrow (\mathsf{K}\alpha \vee \bigvee_{1\le i\le k} \mathsf{K}(\alpha \vee \neg\beta_i)) \qquad \mathsf{S5}_\mathsf{K} \qquad (2)$$
$$\vdash \mathsf{K}(\alpha \vee \mathsf{K}(\alpha \vee \neg\beta_i)) \leftrightarrow (\mathsf{K}\alpha \vee \mathsf{K}(\alpha \vee \neg\beta_i)) \qquad \mathsf{S5}_\mathsf{K} \qquad (3)$$
$$\vdash \mathsf{K}(\alpha \vee \bigvee_{1\le i\le k} \mathsf{K}(\alpha \vee \neg\beta_i)) \to \bigvee_{1\le i\le k} \mathsf{K}(\alpha \vee \mathsf{K}(\alpha \vee \neg\beta_i)) \qquad (1)(2)(3)\mathtt{rRE} \qquad (4)$$
$$\vdash \Box(\widehat{\mathsf{K}}\alpha \vee \bigvee_{1\le i\le k} \mathsf{K}\neg\beta_i) \leftrightarrow (\alpha \vee \bigvee_{1\le i\le k} \mathsf{K}(\alpha \vee \neg\beta_i)) \qquad \mathtt{BK}\vee \qquad (5)$$
$$\vdash \Box(\widehat{\mathsf{K}}\alpha \vee \mathsf{K}\neg\beta_i) \leftrightarrow (\alpha \vee \mathsf{K}(\alpha \vee \neg\beta_i)) \qquad \mathtt{BK}\vee \qquad (6)$$
$$\vdash \mathsf{K}\Box(\mathsf{K}\neg\alpha \to \bigvee_{1\le i\le k} \mathsf{K}\neg\beta_i) \to \bigvee_{1\le i\le k} \mathsf{K}\Box(\mathsf{K}\neg\alpha \to \mathsf{K}\neg\beta_i) \qquad (4)(5)(6)\mathtt{rRE}, \mathsf{S5}_\mathsf{K} \qquad (7)$$
$$\vdash \mathsf{K}\Box(\mathsf{Kh}\neg\alpha \to \bigvee_{1\le i\le k} \mathsf{Kh}\neg\beta_i) \to \bigvee_{1\le i\le k} \mathsf{K}\Box(\mathsf{Kh}\neg\alpha \to \mathsf{Kh}\neg\beta_i) \qquad (7)\mathtt{KKhN}, \mathtt{KhK}, \mathtt{rRE} \qquad (8)$$
$$\vdash \mathsf{K}\Box(\mathsf{Kh}\neg\alpha \to \mathsf{Kh}\bigvee_{1\le i\le k} \neg\beta_i) \to \bigvee_{1\le i\le k} \mathsf{Kh}(\neg\alpha \to \neg\beta_i) \qquad (8)\mathtt{Kh}\vee, \mathtt{Kh}\to, \mathtt{rRE} \qquad (9)$$
$$\vdash \mathsf{Kh}(\neg\alpha \to \bigvee_{1\le i\le k} \neg\beta_i) \to \mathsf{Kh}\bigvee_{1\le i\le k} (\neg\alpha \to \neg\beta_i) \qquad (9)\mathtt{Kh}\to, \mathtt{Kh}\vee, \mathtt{rRE} \qquad (10)$$
$$\vdash \mathsf{Kh}((\neg\alpha \to \bigvee_{1\le i\le k} \neg\beta_i) \to \bigvee_{1\le i\le k} (\neg\alpha \to \neg\beta_i)) \qquad (10)\mathtt{RKh}\to \quad \blacksquare \qquad (11)$$

Note that although KhND and KhKP are very similar semantically as in the proof of Proposition 30, deriving KhKP requires more efforts within our proof system due to the absence of negations in front of β_i, which bridged K and Kh in the above proof. Nevertheless, $\mathtt{KhKP}$ can be proved via a detour using Theorem 54 that gives the equivalent $\bigvee \mathsf{K}\rho_i$ form of each Kh-formula, which then reduces the case to $\mathtt{KhND}$.[5]

To end this section, we discuss the connections of our logic of knowing how with the *planning-based* knowing how logic studied in [15,20,35]. In [20], $\mathsf{Kh}\varphi$ roughly says that there is a plan such that I know that executing it will always guarantee φ, where φ can be any formula in the language. Note that this is very different from the intuitive reading of know-how operator in the current framework, where $\mathsf{Kh}\alpha$ says that knowing how α is true, where α is a propositional formula. The fundamental difference in the semantics is reflected by the axioms regarding the Kh modality. For example, $\mathsf{Kh}(\varphi \vee \psi) \to (\mathsf{Kh}\varphi \vee \mathsf{Kh}\psi)$ is not valid in [15,20], e.g., one can always use any plan to make sure $p \vee \neg p$ but it does not mean one can make sure p or make sure $\neg p$. On the other hand, $\mathsf{K}\varphi \to \mathsf{Kh}\varphi$ is valid in [15,20] since you can always use the empty plan, whereas in our framework, it does not hold in general when φ is not a statement.

5. Epistemic interpretation of inquisitive semantics

In this section, we look at the concepts in inquisitive semantics from our alternative epistemic perspective.

We first summarize the correspondence between our semantics and inquisitive semantics below based on the corresponding definitions in Section 2.[6]

Inquisitive semantics	Our epistemic semantics
information model	single-agent S5 epistemic model (with an implicit total relation)
non-empty states	epistemic submodels
support $(\mathcal{M}, s \Vdash \alpha)$	know-how $(\mathcal{M}_s \vDash \mathsf{Kh}\alpha)$
alternatives for α in $\mathcal{M}$	maximal submodels of $\mathcal{M}$ satisfying $\mathsf{Kh}\alpha$
proposition expressed by α in $\mathcal{M}$	set of submodels of $\mathcal{M}$ for $\mathsf{Kh}\alpha$
α is inquisitive in $\mathcal{M}$	there are two maximal submodels satisfying $\mathsf{Kh}\alpha$
α is informative in $\mathcal{M}$	there is one world not in any maximal submodels of $\mathcal{M}$ for $\mathsf{Kh}\alpha$

First we give an epistemic characterization of informativeness (and questions). The idea is that α is informative iff you do not know that α already.

Proposition 46. *Given any $\mathcal{M}$, α is informative in $\mathcal{M}$ iff $\mathcal{M} \vDash \neg\mathsf{K}\alpha$. Thus α is a question in $\mathcal{M}$ iff $\mathcal{M} \vDash \mathsf{K}\alpha$.*

Proof. Recall that α is informative in $\mathcal{M}$ iff there is at least one world in $\mathcal{M}$ that is not included in any alternative for α in $\mathcal{M}$ (cf. Definition 9). This definition can be rendered intuitively (and formally) as $\mathcal{M} \vDash \widehat{\mathsf{K}}\neg\Diamond\mathsf{Kh}\alpha$ in our framework. Now due to Proposition 28, it is equivalent to $\mathcal{M} \vDash \widehat{\mathsf{K}}\neg\alpha$, namely, $\mathcal{M} \vDash \neg\mathsf{K}\alpha$. ■

Next, we prove an epistemic characterization of inquisitiveness and statements. The idea is that α is inquisitive in s iff it is possible to know α while not knowing how to resolve it with the information provided by s in a model $\mathcal{M}$.

Proposition 47. *Given any $\mathcal{M}$, α is inquisitive in $\mathcal{M}$ iff $\mathcal{M} \vDash \widehat{\mathsf{K}}\Diamond(\mathsf{K}\alpha \wedge \neg\mathsf{Kh}\alpha)$. Thus α is a statement in $\mathcal{M}$ iff $\mathcal{M} \vDash \mathsf{K}\Box(\mathsf{K}\alpha \to \mathsf{Kh}\alpha)$.*

[5] Instead of $\mathtt{EU}_k$, we may try to add an axiom $\widehat{\mathsf{K}}\Diamond(\mathsf{K}\alpha \wedge \neg\mathsf{Kh}\beta) \wedge \widehat{\mathsf{K}}\Diamond(\mathsf{K}\alpha \wedge \neg\mathsf{Kh}\gamma) \to \widehat{\mathsf{K}}\Diamond(\mathsf{K}\alpha \wedge \neg\mathsf{Kh}\beta \wedge \neg\mathsf{Kh}\gamma)$, which captures the possibility of merging two submodels, thus having a direct link with $\mathtt{KhKP}$ (cf. the proof of Proposition 30).

[6] If $V[s] = \wp(\mathbf{P})$ then we have the corresponding *absolute* notions of inquisitiveness and informativeness as in [11].

Proof. Recall that α is inquisitive in $\mathcal{M}$ iff there are at least *two* alternatives for α (cf. Definition 9). In our terms, it means there is some submodel (e.g., the union of those two alternatives) such that $\mathsf{K}\alpha$ holds but there is no uniform resolution. Formally, it amounts to $\mathcal{M} \vDash \widehat{\mathsf{K}}\Diamond(\mathsf{K}\alpha \wedge \neg\mathsf{Kh}\alpha)$. Therefore α is a statement in $\mathcal{M}$ iff α is not inquisitive iff $\mathcal{M} \vDash \neg\widehat{\mathsf{K}}\Diamond(\mathsf{K}\alpha \wedge \neg\mathsf{Kh}\alpha)$ iff $\mathcal{M} \vDash \mathsf{K}\Box(\mathsf{K}\alpha \to \mathsf{Kh}\alpha)$. ■

Note that $\mathsf{K}\Box(\mathsf{K}\alpha \to \mathsf{Kh}\alpha)$ is equivalent to $\mathsf{Kh}(\neg\neg\alpha \to \alpha)$ which is consistent with the characterization of statements in [11]. $\mathsf{K}\Box(\mathsf{K}\alpha \to \mathsf{Kh}\alpha)$ says that I know that if you tell me α is resolvable then I will know how to resolve it.

In particular, when taking a full model $\mathcal{M}$ and the trivial state $W_{\mathcal{M}}$, then from Propositions 6, 46 and 47, we have the characterization of absolute questions and statements.

Proposition 48. *α is a question iff $\mathsf{K}\alpha$ is valid iff α is a classical tautology. α is a statement iff $\mathsf{K}\alpha \to \mathsf{Kh}\alpha$ is valid iff $\mathsf{K}\alpha \leftrightarrow \mathsf{Kh}\alpha$ is valid.*

Based on the above epistemic characterization of the relative notion of statements, we can prove the following from a purely epistemic perspective extending the result in Proposition 2.19 in [11] about absolute statements.

Proposition 49. *For any $p \in \mathbf{P}$ and $\alpha, \beta \in \mathbf{PL}$, any given model $\mathcal{M}$:*

1. *p is a statement in $\mathcal{M}$.*
2. *$\bot$ is a statement in $\mathcal{M}$.*
3. *If α and β are statements in $\mathcal{M}$, then $\alpha \wedge \beta$ is a statement in $\mathcal{M}$.*
4. *if β is a statement in $\mathcal{M}$, then $\alpha \to \beta$ is a statement in $\mathcal{M}$.*

Proof. (1) and (2) are trivial. For (3): Suppose α and β are statements in $\mathcal{M}$, by Proposition 47, we need to show $\mathcal{M} \vDash \mathsf{K}\Box(\mathsf{K}(\alpha \wedge \beta) \to \mathsf{Kh}(\alpha \wedge \beta))$, which is equivalent to $\mathcal{M} \vDash \mathsf{K}\Box((\mathsf{K}\alpha \wedge \mathsf{K}\beta) \to (\mathsf{Kh}\alpha \wedge \mathsf{Kh}\beta))$. By the assumption that α and β are statements in $\mathcal{M}$, it can be easily proved.

Now for (4): Suppose that β is a statement in $\mathcal{M}$. By Proposition 47, we have $\mathcal{M} \vDash \mathsf{K}\Box(\mathsf{K}\beta \to \mathsf{Kh}\beta)$ (†). We need to show that $\mathcal{M} \vDash \mathsf{K}\Box(\mathsf{K}(\alpha \to \beta) \to \mathsf{Kh}(\alpha \to \beta))$, namely, for any submodel $\mathcal{M}'$ of $\mathcal{M}$ if $\mathcal{M}' \vDash \mathsf{K}(\alpha \to \beta)$ then $\mathcal{M}' \vDash \mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$. Suppose $\mathcal{M}' \vDash \mathsf{K}(\alpha \to \beta)$ and take any submodel $\mathcal{M}''$ of $\mathcal{M}'$ such that $\mathcal{M}'' \vDash \mathsf{Kh}\alpha$, we only need to show $\mathcal{M}'' \vDash \mathsf{Kh}\beta$. Since $\vDash \mathsf{Kh}\alpha \to \mathsf{K}\alpha$, we have $\mathcal{M}'' \vDash \mathsf{K}\alpha$. Since $\mathcal{M}' \vDash \mathsf{K}(\alpha \to \beta)$ and $\mathcal{M}''$ is a submodel of $\mathcal{M}'$, we have $\mathcal{M}'' \vDash \mathsf{K}\beta$. Now due to the fact that $\mathcal{M}''$ is also a submodel of $\mathcal{M}$, we have $\mathcal{M}'' \vDash \mathsf{Kh}\beta$ by (†), which completes the proof. ■

As an immediate consequence, any disjunction-free formula is a statement in any $\mathcal{M}$.

Proposition 50. *For any $\alpha \in \mathbf{PL}$, if α is disjunction-free, then α is a statement in $\mathcal{M}$ for any $\mathcal{M}$.*

Next, we show the epistemic proof of another result in [11] linking the three most important concepts in inquisitive semantics.

Proposition 51 *(Support, inquisitiveness, and informativeness). $\mathcal{M}, s$ supports a formula α if α is neither inquisitive in $\mathcal{M}_s$ nor informative in $\mathcal{M}_s$.*

Proof. It is straightforward in our case due to the validity of $\mathsf{Kh}\alpha \leftrightarrow (\mathsf{K}\alpha \wedge \mathsf{K}\Box(\mathsf{K}\alpha \to \mathsf{Kh}\alpha))$. The right-to-left implication is due to the validity of the axioms $\mathtt{T}_{\Box}$ and $\mathtt{T}_{\mathsf{K}}$. The left-to-right implication is due to the validity of `KhK` and the fact that once we have a uniform resolution for α then we still have it in any submodel. ■

As we mentioned earlier, the intended interpretation of inquisitive logic is not about knowledge, and what we are presenting is an alternative interpretation. On the other hand, there are intimate connections between the two interpretations conceptually. We conclude the section by the following discussion on knowledge and information range. Starting from Hintikka [17], knowledge in epistemic logic is defined based on information range: φ is known iff in φ holds on all the epistemic alternatives of the real world. This notion of knowledge is applied to various fields where the object to which we ascribe knowledge statements is not an agent at all, such as in distributed system of computer science. Note that in many applications, the so-called "agent" is just a way of talking about the information range. It is also reflected in the technical fact that the classical propositional reasoning can be cast in epistemic logic such that $\alpha \vDash_{\mathbf{CPL}} \beta$ iff $\mathsf{K}\alpha \vDash_{\mathbf{S5}} \mathsf{K}\beta$. Intuitively we can turn a classical propositional entailment from α to β into an epistemic entailment by *if α is known, then β is also known*. As we have seen in Theorem 35, we can generalize it to match the entailment of **InqB** with then entailment in our know-how variant of epistemic logic. Moreover, when talking about general knowledge, it is not essential to ask which specific agent the knowledge is ascribed to. Actually, the basic systems of epistemic logic are suitable for this kind of general reasoning, by assuming an idealized reasoner, no matter *who* exactly knows. From our perspective, when we do not give any special constraints on the information that the agent has, we can actually talk about the general semantic knowledge which is not ascribed to a particular agent. Therefore, to us, there is no drastic conceptual gap between the information model for the inquisitive semantics and epistemic model in our setting. Nevertheless, in concrete applications about real agents, some more detailed constraints on what agents know and what they can learn may be given, which may result in changes of logic (cf. e.g., [9, Section 2.8]).

6. Related work

In this section, we first connect the formula-based resolutions studied in the literature of inquisitive logic in [8,11] to our approach, and then compare our work to the related work on modal logic in the literature.

6.1. Resolutions in terms of formulas in PL

In our framework, the resolutions defined in Definitions 12 and 14 are not in the object language. It is not hard to check as long as $S(p)$ is a singleton $\{x\}$ and $R(w,p) = S(p)$ iff $p \in V_{\mathcal{M}}(w)$, the semantics will not affect the logic.[7] On the other hand, since the resolutions of atomic propositions are defined simply as themselves, it is possible to express the resolutions using formulas in the object language. In [11], the authors proposed a notion of resolutions expressed by disjunction-free formulas in **PL**. Here we take the simplified version Definition 8 in [8][8]:

Definition 52 *(Resolutions [8]).*

- $RL(p) = \{p\}$ for $p \in \mathbf{P}$
- $RL(\bot) = \{\bot\}$
- $RL(\alpha \vee \beta) = RL(\alpha) \cup RL(\beta)$
- $RL(\alpha \wedge \beta) = \{\rho \wedge \sigma \mid \rho \in RL(\alpha) \text{ and } \sigma \in RL(\beta)\}$
- $RL(\alpha \to \beta) = \{\bigwedge_{\rho \in RL(\alpha)} (\rho \to f(\rho)) \mid f : RL(\alpha) \to RL(\beta)\}$

For example, instead of using the explicit functions for the resolutions of $\alpha \to \beta$, RL uses a conjunction of implications to capture the function. As proved in [11], each element of $R(\alpha)$ is a *statement*.

[7] Note that what exactly *is* the resolution of each atomic proposition does not matter, due to our BHK-like semantics for Kh, which only checks the existence of (uniform) resolutions.

[8] In Definition 9.1 in [11], the resolution formulas (realizations) are based on some normal form.

The following proposition is stated without a proof in Proposition 9.3 in [11], which can be shown by an inductive proof.

Proposition 53 *([11]). For any model $\mathcal{M}$, state s and formula $\alpha \in \mathbf{PL}$,*

$$\mathcal{M}, s \Vdash \alpha \iff s \subseteq |\rho|_{\mathcal{M}} \text{ for some } \rho \in RL(\alpha)$$

where $|\rho|_{\mathcal{M}}$ is the set of possible worlds that satisfy ρ classically in $\mathcal{M}$.

In our know-how perspective, the above proposition actually establishes the equivalence of the state-based inquisitive semantics to the following alternative epistemic semantics for $\mathsf{Kh}\alpha$.

$$\boxed{\mathcal{M}, w \Vdash\!\vDash \mathsf{Kh}\alpha \iff \text{there exists a } \rho \in RL(\alpha) \text{ such that } \mathcal{M}, w \vDash \mathsf{K}\rho}$$

We can establish the following equivalences, without using Proposition 53.

Theorem 54. *For any model $\mathcal{M}$ and any non-empty state s, and any $\alpha \in \mathbf{PL}$, the following are equivalent:*

1. $\mathcal{M}, s \Vdash \alpha$
2. $\mathcal{M}_s \vDash \mathsf{Kh}\alpha$
3. $\mathcal{M}_s \vDash \bigvee_{\rho \in RL(\alpha)} \mathsf{K}\rho$
4. $\mathcal{M}_s \Vdash\!\vDash \mathsf{Kh}\alpha$.

Proof. (1) iff (2) is due to Proposition 34. (3) iff (4) is based on the definition of $\Vdash\!\vDash$ and the fact that $RL(\alpha)$ is finite. In the following, we show that (2) iff (3) by induction on the structure of α, where the definition of RL and classical reasoning play an important role.

- $\alpha = p$ or $\alpha = \bot$: It is obvious since $RL(\alpha) = \{\alpha\}$ and $\vDash \mathsf{Kh}\alpha \leftrightarrow \mathsf{K}\alpha$ in such cases.
- $\alpha = \alpha_1 \vee \alpha_2$: $\mathcal{M}_s \vDash \mathsf{Kh}(\alpha_1 \vee \alpha_2) \iff \mathcal{M}_s \vDash \mathsf{Kh}\alpha_1 \vee \mathsf{Kh}\alpha_2 \iff \mathcal{M}_s \vDash \bigvee_{\rho \in RL(\alpha_1)} \mathsf{K}\rho \vee \bigvee_{\rho \in RL(\alpha_2)} \mathsf{K}\rho \iff \mathcal{M}_s \vDash \bigvee_{\rho \in RL(\alpha)} \mathsf{K}\rho$
- $\alpha = \alpha_1 \wedge \alpha_2$: $\mathcal{M}_s \vDash \mathsf{Kh}(\alpha_1 \wedge \alpha_2) \iff \mathcal{M}_s \vDash \mathsf{Kh}\alpha_1 \wedge \mathsf{Kh}\alpha_2 \iff \mathcal{M}_s \vDash \bigvee_{\rho \in RL(\alpha_1)} \mathsf{K}\rho \wedge \bigvee_{\rho \in RL(\alpha_2)} \mathsf{K}\rho \iff \mathcal{M}_s \vDash \bigvee_{\rho_1 \in RL(\alpha_1), \rho_2 \in RL(\alpha_2)} \mathsf{K}\rho_1 \wedge \mathsf{K}\rho_2 \iff \mathcal{M}_s \vDash \bigvee_{\rho_1 \in RL(\alpha_1), \rho_2 \in RL(\alpha_2)} \mathsf{K}(\rho_1 \wedge \rho_2) \iff \mathcal{M}_s \vDash \bigvee_{\rho \in RL(\alpha)} \mathsf{K}\rho$
- $\alpha = \alpha_1 \to \alpha_2$: $\mathcal{M}_s \vDash \mathsf{Kh}(\alpha_1 \to \alpha_2) \iff \mathcal{M}_s \vDash \mathsf{K}\Box(\mathsf{Kh}\alpha_1 \to \mathsf{Kh}\alpha_2)$
  Based on the induction hypothesis and the definition of $RL(\alpha \to \beta)$ we just need to prove that:

$$\mathcal{M}_s \vDash \mathsf{K}\Box(\bigvee_{\rho \in RL(\alpha_1)} \mathsf{K}\rho \to \bigvee_{\rho \in RL(\alpha_2)} \mathsf{K}\rho) \iff \mathcal{M}_s \vDash \bigvee_{f: RL(\alpha_1) \to RL(\alpha_2)} \mathsf{K} \bigwedge_{\rho \in RL(\alpha_1)} (\rho \to f(\rho)).$$

  $\Longrightarrow$: Given $\mathcal{M}_s \vDash \mathsf{K}\Box(\bigvee_{\rho \in RL(\alpha_1)} \mathsf{K}\rho \to \bigvee_{\rho \in RL(\alpha_2)} \mathsf{K}\rho)$, we need to find a $f : RL(\alpha_1) \to RL(\alpha_2)$ s.t. $\mathcal{M}_s \vDash \mathsf{K} \bigwedge_{\rho \in RL(\alpha_1)} (\rho \to f(\rho))$. As $RL(\alpha_2)$ is not empty, let ρ_0 be a fixed element of $RL(\alpha_2)$ to be used to define the function f. There are two cases to be considered.
  - For any $\rho \in RL(\alpha_1)$ such that $\mathcal{M}_s \vDash \mathsf{K}\neg\rho$, we have $\mathcal{M}_s \vDash \mathsf{K}(\rho \to \rho')$ for any $\rho' \in RL(\alpha_2)$. We can safely define $f(\rho) = \rho_0$.
  - For any $\rho \in RL(\alpha_1)$ such that $\mathcal{M}_s \vDash \widehat{\mathsf{K}}\rho$, let $\mathcal{M}_s^\rho$ be the maximal submodel of $\mathcal{M}_s$ s.t. $\mathcal{M}_s^\rho \vDash \mathsf{K}\rho$. By the semantics of K and $\Box$, if $\mathcal{M}_s \vDash \mathsf{K}\Box(\bigvee_{\rho \in RL(\alpha_1)} \mathsf{K}\rho \to \bigvee_{\rho \in RL(\alpha_2)} \mathsf{K}\rho)$, $\mathcal{M}_s^\rho \vDash \bigvee_{\rho \in RL(\alpha_1)} \mathsf{K}\rho \to \bigvee_{\rho \in RL(\alpha_2)} \mathsf{K}\rho$, then $\mathcal{M}_s^\rho \vDash \bigvee_{\rho \in RL(\alpha_2)} \mathsf{K}\rho$, which means that there is a $\rho' \in RL(\alpha_2)$ such that $\mathcal{M}_s^\rho \vDash \mathsf{K}\rho'$. As $\mathcal{M}_s^\rho$ is the maximal submodel of $\mathcal{M}_s$ s.t. $\mathcal{M}_s^\rho \vDash \mathsf{K}\rho$, $\mathcal{M}_s \vDash \mathsf{K}(\rho \to \rho')$. We let $f(\rho) = \rho'$.

Now we have defined an $f : RL(\alpha_1) \to RL(\alpha_2)$ s.t. $\mathcal{M}_s \vDash \bigwedge_{\rho\in RL(\alpha_1)} \mathsf{K}(\rho \to f(\rho))$, which is equivalent to $\mathcal{M}_s \vDash \mathsf{K} \bigwedge_{\rho\in RL(\alpha_1)}(\rho \to f(\rho))$.
$\Longleftarrow$: Suppose $\mathcal{M}_s \vDash \bigvee_{f:RL(\alpha_1)\to RL(\alpha_2)} \mathsf{K} \bigwedge_{\rho\in RL(\alpha_1)}(\rho \to f(\rho))$ then there is a $f : RL(\alpha_1) \to RL(\alpha_2)$ s.t. $\mathcal{M}_s \vDash \mathsf{K} \bigwedge_{\rho\in RL(\alpha_1)}(\rho \to f(\rho))$. This amounts to $\mathcal{M}_s \vDash \bigwedge_{\rho\in RL(\alpha_1)} \mathsf{K}(\rho \to f(\rho))$, thus for any submodel $\mathcal{M}'$ of $\mathcal{M}_s$, $\mathcal{M}' \vDash \bigwedge_{\rho\in RL(\alpha_1)}(\mathsf{K}(\rho \to f(\rho)))$. By the usual distribution axiom by of K, $\mathcal{M}' \vDash \bigwedge_{\rho\in RL(\alpha_1)}(\mathsf{K}\rho \to \mathsf{K}f(\rho))$. Weakening the consequent, we have $\mathcal{M}' \vDash \bigwedge_{\rho\in RL(\alpha_1)}(\mathsf{K}\rho \to \bigvee_{\rho\in RL(\alpha_2)} \mathsf{K}\rho)$. Therefore $\mathcal{M}' \vDash \bigvee_{\rho\in RL(\alpha_1)} \mathsf{K}\rho \to \bigvee_{\rho\in RL(\alpha_2)} \mathsf{K}\rho$. Since $\mathcal{M}'$ is an arbitrary submodel of $\mathcal{M}_s$, it follows that $\mathcal{M}_s \vDash \mathsf{K}\Box(\bigvee_{\rho\in RL(\alpha_1)} \mathsf{K}\rho \to \bigvee_{\rho\in RL(\alpha_2)} \mathsf{K}\rho)$. ■

As an immediate consequence, we have the modal translation from **InqB** to the epistemic logic **S5** and the modal logic **K**. The translation to modal logic **K** (and other normal modal logics) was mentioned by Ciardelli in [7, Section 6.6], and also in [8, Section 5.4] when discussing the modal approach of [26] to the semantics of questions. Through the correspondence with the know-how semantics (w.r.t. $\Vdash$) shown in the above theorem, we can see more intuitively what the translation below is doing.

Corollary 55 *(**InqB** to **S5** and **K** [7,8]). For $\alpha \in \mathbf{PL}$, $\alpha \in \mathbf{InqB}$ iff $\bigvee_{\rho\in RL(\alpha)} \mathsf{K}\rho \in \mathbf{S5}$ iff $\bigvee_{\rho\in RL(\alpha)} \mathsf{K}\rho \in \mathbf{K}$.*

Proof. The last *iff* is due to the fact that for each pointed Kripke model $\mathcal{M}, w \vDash \neg\bigvee_{\rho\in RL(\alpha)} \mathsf{K}\rho$ there is an epistemic model $\mathcal{N}, w \vDash \neg\bigvee_{\rho\in RL(\alpha)} \mathsf{K}\rho$ where $\mathcal{N} = \langle W_{\mathcal{M}}, \sim, V_{\mathcal{M}}\rangle$ and $\sim$ is the reflexive, symmetric, transitive closure of the accessibility relation in $\mathcal{M}$. Note that $\neg\bigvee_{\rho\in RL(\alpha)} \mathsf{K}\rho$ is equivalent to $\bigwedge_{\rho\in RL(\alpha)} \widehat{\mathsf{K}}\neg\rho$. ■

Note that although it is natural to think this translation can be compared to Gödel's translation of intuitionistic logic to $\mathsf{S4}$, the nature of the two translations is quite different in terms of how to read the modality. In Gödel's translation, the modality is actually a temporal-epistemic one, namely $\mathsf{K}\Box$ in our perspective [37], but the modality here is merely a purely epistemic one.

As another corollary of Theorem 54, the following important property of **InqB** follows.

Corollary 56 *([6,11]). Any $\alpha \in \mathbf{PL}$ is equivalent to a disjunction of statements/negations in inquisitive logic.*

Proof. From Theorem 54, $\vDash \mathsf{Kh}\alpha \leftrightarrow \bigvee_{\rho\in RL(\alpha)} \mathsf{K}\rho$. Due to Proposition 50, $\rho \in RL(\alpha)$ is a statement. Therefore by Proposition 48, $\vDash \mathsf{Kh}\alpha \leftrightarrow \bigvee_{\rho\in RL(\alpha)} \mathsf{Kh}\rho$, thus $\vDash \mathsf{Kh}\alpha \leftrightarrow \mathsf{Kh}\bigvee_{\rho\in RL(\alpha)} \rho$. By the validity of the Rule $\mathtt{RKh}\!\to$, we have $\vDash \mathsf{Kh}(\alpha \leftrightarrow \bigvee_{\rho\in RL(\alpha)} \rho)$. By Corollary 36, we have $\alpha \leftrightarrow \bigvee_{\rho\in RL(\alpha)} \rho \in \mathbf{InqB}$. ■

The above corollaries show that inquisitive logic can be viewed as a fragment of normal epistemic logic technically. However, as argued in [8] regarding the modal approach of [26], those modal formulas do not preserve the surface structure of sentences and questions in the natural language. Actually, as shown by our approach, such a reduction to standard epistemic logic is the result of the assumption in inquisitive logic that each atomic proposition has a unique resolution. In similar settings such as intuitionistic logic and Medvedev logic, this is not possible. Using our powerful language **DELKh**, we can keep the structure of statements and questions as they are in the natural language by $\mathsf{Kh}\alpha$ on the one hand, and reveal its epistemic meaning by the reductions on the other hand. We do not need to take sides between the technical and conceptual convenience.

Yet another consequence of Theorem 54 is about the limitation of inquisitive logic over models.

Corollary 57. *Inquisitive logic is less expressive than* **EL**.

Proof. Note that inquisitive logic can only say things in terms of disjunction over positive $\mathsf{K}\alpha$ formulas, thus formulas like $\widehat{\mathsf{K}}p$ are simply not expressible. ■

With the classical connectives in hand, we can express various things which were not expressible in the standard inquisitive logic, such as the classical negation of an inquisitive formula α (simply by $\neg \mathsf{Kh}\alpha$) (cf. [28] for the study within the framework of inquisitive logic). The classical connectives and normal know-that modality give us a lot of flexibility in capturing mixed reasoning with both inquisitive and classical propositions.

6.2. Comparison with inquisitive modal logic

In the literature, *Inquisitive Modal Logic* is proposed as a conservative extension of normal modal logic by introducing the modality in the language and separating the inquisitive disjunction $\vee\!\!\!\vee$ with the classical disjunction $\vee$ (cf. e.g., [7]). To avoid potential confusion with our $\Box$ modality, we denote the inquisitive modality as $\boxdot$ in this subsection. With $\boxdot$ at hand, one can naturally express formulas combining the modality and questions, such as $\boxdot(p \vee\!\!\!\vee \neg p)$, which says *knowing whether* p in an epistemic setting. Correspondingly, the information model is extended with a binary relation $\mathcal{R}$, not to be confused with the resolution function R, to interpret the modality with the following extra semantic clause in Definition 6.1.3 in [7]:

$$\boxed{\mathcal{M}, s \Vdash \boxdot\alpha \quad \Longleftrightarrow \quad \text{for all } w \in s,\ \mathcal{M}, \mathcal{R}[w] \Vdash \alpha}$$

where $\mathcal{R}[w] = \{v \mid w\mathcal{R}v \text{ in } \mathcal{M}\}$. When s is a singleton set, we can derive the following semantics for $\boxdot$ over *worlds* (Proposition 6.1.4 in [7]):

$$\boxed{\mathcal{M}, w \Vdash \boxdot\alpha \quad \Longleftrightarrow \quad \mathcal{M}, \mathcal{R}[w] \Vdash \alpha}$$

By the above support-based semantics, $\boxdot\alpha$ is always *truth-conditional*, i.e., s supports $\boxdot\alpha$ iff it is true at each world $w \in s$ (cf. [7]). When α is also truth-conditional, the above semantics of $\boxdot\alpha$ over worlds clearly boils down to the Kripke semantics for $\boxdot$: $\boxdot\alpha$ is true at w if α is true at each v accessible from w. In [7], a general method of axiomatizating the logic is provided that works with various frame conditions.

Since this is another way of combining modalities and inquisitive connectives, it deserves a comparison with our approach, in particular about the semantics of $\boxdot$ under an epistemic reading and our Kh.

By definition, $\boxdot$ relies on the support relation, while Kh has the $\exists\mathsf{K}$-shape semantics that relies on the resolutions as in Definition 14. However, they are connected deeply and come down to the same truth condition on pointed models. To see this, first let us consider the special case where $\mathcal{R}$ is the total-relation in a model $\mathcal{M}$ for inquisitive modal logic. Therefore, for any $w \in W_{\mathcal{M}}$, $\mathcal{R}[w]$ is exactly the set of possible worlds $W_{\mathcal{M}}$. Given $\mathcal{M}$, let s_t be the trivial state $W_{\mathcal{M}}$, and let $w \in s_t$, the support semantics of $\boxdot\alpha$ over s_t collapses into the one for the non-modal α:

$$\boxed{\mathcal{M}, s_t \Vdash \boxdot\alpha \quad \Longleftrightarrow \quad \mathcal{M}, s_t \Vdash \alpha \Longleftrightarrow \mathcal{M}, w \Vdash \boxdot\alpha}$$

Note that by Lemma 33, we have:

$$\boxed{\mathcal{M}, s_t \Vdash \alpha \quad \Longleftrightarrow \quad \mathcal{M} \vDash \mathsf{Kh}\alpha \Longleftrightarrow \mathcal{M}, w \vDash \mathsf{Kh}\alpha}$$

From the above two observations, we can establish that for any $\alpha \in \mathbf{PL}$:

$$\boxed{\mathcal{M}, w \Vdash \boxdot\alpha \quad \Longleftrightarrow \quad \mathcal{M}, w \vDash \mathsf{Kh}\alpha}$$

The above equivalence still holds on models with an arbitrary relation $\mathcal{R}$,[9] not just the total one, given that we generalize our Kh semantics naturally to models with $\mathcal{R}$, as in logics of *know-wh* (cf. e.g., [36]).

[9] It is observed by the anonymous reviewer.

$$\boxed{\mathcal{M}, w \models \mathsf{Kh}\alpha \iff \text{there exists an } x \text{ s.t. for any } v \in \mathcal{R}[w], x \in R(v, \alpha)}$$

Under this more general semantics of Kh over models with an arbitrary relation $\mathcal{R}$, we can establish:

$$\boxed{\mathcal{M}, w \Vdash \boxdot\alpha \iff \mathcal{M}, R[w] \Vdash \alpha \iff \mathcal{M}, w \models \mathsf{Kh}\alpha}$$

The second equivalence is again an application of Lemma 33 under the help of Proposition 32. Essentially, the equivalence is due to the fact that the intended function of Kh is exactly to capture the support relations between a state and an inquisitive formula, and the truth conditional semantics of $\boxdot$ gave $\mathcal{R}[w]$ as such a state. Thus the two different routes that Kh and $\boxdot$ take in their apparent differently semantics converge to the same truth condition eventually. This equivalence is also suggested by Proposition 53 and Theorem 54 showing that the support semantics can be viewed as a resolution-based semantics. However, in general, it is not guaranteed that the (re)solution-based semantics, such as the one for Medvedev's logic, can be transformed into an equivalent state-based semantics over information models, for the structure of resolutions for (atomic) propositions is richer than the mere truth values on each world. In the case of inquisitive logic, the situation is very much simplified by assuming the atomic propositions have one and only possible resolution.

There is also an interesting parallel between our work and inquisitive modal logic. In our setting, we explicitly separate the different roles of the *modalities* by using both K and Kh. On the other hand, we use the same symbol $\vee$ for inquisitive and classical *disjunctions*. In contrast, in inquisitive modal logic, the modality has a double role to play depending on what is in the scope, while the two disjunctions are differentiated explicitly by distinct symbols.

The difference is firstly conceptual. From our epistemic perspective, the non-classical behavior of logical connectives is due to the implicit epistemic modality Kh. From the more recent perspective of inquisitive logic [7], the separation of the two disjunctions makes it more clear that the inquisitive logic can be viewed as an extension of classical logic with the inquisitive operators. The difference also leads to various logical properties. For example, since we use the same symbol $\vee$ for these two disjunctions, it can function differently inside and outside the scope of Kh. It follows that the usual unconditional rule of *replacement of equals* in the scope of a Kh modality is invalid, thus making Kh a *hyperintensional operator*. On the other hand, by having K and Kh explicitly, we can differentiate them in the axioms and reveal how negation and atomic propositions can act as bridges to connect the two types of knowledge, as in $\mathsf{K}p \leftrightarrow \mathsf{Kh}p$ and $\mathsf{Kh}\neg\alpha \leftrightarrow \mathsf{K}\neg\alpha$. Note that, thanks to our choice of using the same $\vee$ symbol, the axioms such as $\mathsf{Kh}\alpha \to \mathsf{K}\alpha$ can be written in a natural way, without introducing unnecessary translations of formulas. We think each approach has its features and advantages. In particular, the inquisitive modality has the very elegant feature of deriving the semantics of statements in terms of *know+embedded* questions compositionally (cf. [7, Section 6.2]). The combination of the two approaches can be explored in the future.

Beyond the above connections and differences, Kh and $\boxdot$ are very different in the motivation behind them. The point of our Kh operator is to capture the epistemic content *already* in the support semantics, at least from our epistemic perspective, while the modality in inquisitive modal logic is to *add* the modal information into the picture. More importantly, in our framework, further modalities and connectives are used to "open up" the Kh formulas to reveal their intuitive epistemic readings. In a nutshell, we want to turn the inquisitive formulas into classical ones with also epistemic and dynamic modalities to obtain intuitive epistemic readings of them. For example, our approach also features a dynamic modality $\Box$ to open up $\mathsf{Kh}(\alpha \to \beta)$ by the equivalent $\mathsf{K}\Box(\mathsf{Kh}\alpha \to \mathsf{Kh}\beta)$, i.e., knowing how α implies β means knowing that whenever one knows how α is true, one also knows how β is true. All these extra modalities and the classical connectives helped us to "decode" the non-classical behaviors of the inquisitive logic, from our epistemic perspective. As our intuitive axioms showed, the information behind the innocent-looking propositional formulas of inquisitive logic is very rich, under the epistemic view of point. Interestingly, as we also showed in the paper, $\Box$ and Kh can be eliminated eventually.

There is one more distinction to be pointed out. In our setting, the modalities cannot occur in the scope of Kh operators, but this may not be an essential restriction (at least for the modality K), given the discussion on the resolutions for inquisitive modalities (cf. [7, Section 6.3]). We leave the study of the extended language for a future occasion.[10]

As another incarnation of inquisitive modal logic with dynamics, *Inquisitive Dynamic Epistemic Logic* (**InqDEL**) is proposed and studied in [12,32]. Since our approach is also *dynamic-epistemic* in nature, it also deserves some comparison with **InqDEL**. Again, our approach is to reveal the dynamic-epistemic structure implicitly in the existing inquisitive logic from the epistemic perspective, whereas **InqDEL** extends the version of inquisitive semantics with extra epistemic structures and dynamics. In the models of **InqDEL**, there is an *issue function* Σ assigning each possible world an *issue* $\Sigma(w)$, i.e., a non-empty, downward closed set of states, satisfying some intuitive epistemic conditions. At each world w, the set of epistemically indistinguishable worlds $\sigma(w)$ is then defined as $\bigcup\Sigma(w)$. Moreover, the work of [12] is based on a specific version of inquisitive semantics in [13] where a dichotomous syntax, distinguishing declarative and interrogative sentences is used instead of the unified framework of [11]. On top of this dichotomous language, the know-that modality and an extra modality of *entertain* are added, where the latter modality can describe the issue currently in concern.[11] In contrast, as we mentioned, the dynamic operator $\Box$ in our setting is merely to capture the inquisitive implication. In the model, we also do not have the structures of issues.

7. Conclusions

This paper is a case study of the general research program proposed in [37] on "epistemicizating" intuitionistic logic and its relatives. We showed that, as an alternative interpretation, the propositional inquisitive logic **InqB** can be viewed as a (dynamic) epistemic logic of knowing how over standard S5 epistemic models. In our approach, an inquisitive formula α being supported by a state s is formalized as it is known how to resolve α (or simply knowing how α is true). We start by turning an inquisitive formula α into the equivalent know-how formula $\mathsf{Kh}\alpha$ in our framework. Then by using modalities of know-that K and informational updates $\Box$ based on classical connectives, we can *unload* the epistemic contents hidden in such know-how formulas $\mathsf{Kh}\alpha$ by reducing the complexity of α step by step. From the point of view of the general program of [37], **InqB** is a particularly interesting case since the corresponding know-how modality can be eventually eliminated based on the fact that the resolution of each atomic proposition is *unique*, which is the reason that the axiom $\neg\neg p \to p$ holds for atomic propositions p in inquisitive logic. In our framework, it amounts to the crucial axiom $\mathsf{K}p \to \mathsf{Kh}p$, i.e., *knowing that* p is true implies the apparently stronger *knowing how* it is true, which can help to reduce the know-how operator eventually. Given such a simplification, technically, we can view inquisitive logic as a fragment of standard epistemic logic.

What we have presented so far is clearly only the beginning of an interesting story regarding the classical *"epistemicization"* of the intuitionistic logic and its friends. Here we just list a few further directions.

- Given the close connections between inquisitive logic and dependence logic (cf. e.g., [6]), it is definitely interesting to see how we can give epistemic interpretations of dependence logic in various forms, where in the semantics a *team* can be viewed as an epistemic model. Note that as observed by [10], the crucial notion of disjunction in dependence logic, i.e., the tensor, is not directly definable in inquisitive logic. This also presents a challenge to the epistemicization of dependence logic in our framework since we

[10] We also thank the anonymous reviewer for pointing out this to us.
[11] A similar dynamic epistemic approach handling issues is [31] (cf. [12] for a detailed comparison between [12] and [31]).

would like to unload the epistemic content of tensor disjunction in a compositional manner. See [33] for the first attempt.

- As in intuitionistic logic, quantifiers may bring extra complications, so it is interesting to see how we can extend our work to capture the first-order inquisitive logic [4].
- Our approach also opens a natural way to extend inquisitive logic to a multi-agent setting. However, our reduction of the dynamic operator relies on the single-agent setting. It remains to see whether we can still reduce the multi-agent epistemicization of inquisitive logic to the multi-agent S5.
- The interpretation of inquisitive formulas in our setting as *knowing how* α *is true* has an obvious connection with truthmaker semantics based on the intuitionistic spirit [16]. The exact connections invite close investigations.
- It is also interesting to see how we can combine the idea of inquisitive modal logics with our work to benefit from both frameworks.

Declaration of competing interest

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Appendix A. Remaining proof for Proposition 38

Proof. In the following, let $\mathtt{S5_K}$ be the proof system of **S5** including $\mathtt{T_K}, \mathtt{4_K}$, and $\mathtt{5_K}$. For $\mathtt{B}\vee$:

$\vdash \alpha \leftrightarrow \Box\alpha$	$\mathtt{Per}, \mathtt{T}_\Box$,	(1)
$\vdash \alpha \vee \Box\varphi \leftrightarrow \Box\alpha \vee \Box\varphi$	(1)$\mathtt{rRE}$	(2)
$\vdash \Box\alpha \vee \Box\varphi \to \Box(\alpha \vee \varphi)$	$\mathtt{S4}_\Box$	(3)
$\vdash \alpha \vee \Box\varphi \to \Box(\alpha \vee \varphi)$	(2)(3)$\mathtt{rRE}$	(4)
$\vdash \Box(\alpha \vee \varphi) \to (\alpha \wedge \Box(\alpha \vee \varphi)) \vee (\neg\alpha \wedge \Box(\alpha \vee \varphi))$	$\mathtt{TAUT}$	(5)
$\vdash \neg\alpha \leftrightarrow \Box\neg\alpha$	$\mathtt{Per}, \mathtt{T}_\Box$	(6)
$\vdash (\neg\alpha \wedge \Box(\alpha \vee \varphi)) \leftrightarrow (\Box\neg\alpha \wedge \Box(\alpha \vee \varphi))$	(6)$\mathtt{rRE}$	(7)
$\vdash (\Box\neg\alpha \wedge \Box(\alpha \vee \varphi)) \leftrightarrow \Box(\neg\alpha \wedge (\alpha \vee \varphi))$	$\mathtt{S4}_\Box$	(8)
$\vdash \Box(\neg\alpha \wedge (\alpha \vee \varphi)) \leftrightarrow (\Box\neg\alpha \wedge \Box\varphi)$	$\mathtt{S4}_\Box$	(9)
$\vdash (\neg\alpha \wedge \Box(\alpha \vee \varphi)) \to \Box\varphi$	(7)(8)(9)$\mathtt{TAUT}$	(10)
$\vdash \Box(\alpha \vee \varphi) \to (\alpha \wedge \Box(\alpha \vee \varphi)) \vee \Box\varphi$	(5)(10)$\mathtt{S4}_\Box$	(11)
$\vdash \Box(\alpha \vee \varphi) \to \alpha \vee \Box\varphi$	(11)$\mathtt{TAUT}$	(12)
$\vdash \Box(\alpha \vee \varphi) \leftrightarrow \alpha \vee \Box\varphi$	(4)(12)$\mathtt{TAUT}$	(13)

For $\mathtt{BK}\vee$: Let φ be $(\widehat{\mathsf{K}}\alpha \vee \mathsf{K}\alpha_1 \vee \cdots \vee \mathsf{K}\alpha_n)$ and let ψ be $(\alpha \vee \mathsf{K}(\alpha \vee \alpha_1) \vee \cdots \vee \mathsf{K}(\alpha \vee \alpha_n))$ below.

$$\vdash \alpha \to \Box\alpha \qquad \mathtt{Per} \tag{1}$$

$$\vdash \Box\alpha \to \Box\widehat{\mathsf{K}}\alpha \qquad \mathsf{S5_K}, \mathsf{S4_\Box} \tag{2}$$

$$\vdash \alpha \to \Box\widehat{\mathsf{K}}\alpha \qquad (1)(2)\mathtt{TAUT} \tag{3}$$

$$\vdash \alpha \to \Box\varphi \qquad (3)\mathsf{S5_K}, \mathsf{S4_\Box} \tag{4}$$

$$\vdash \alpha \to (\psi \to \Box\varphi) \qquad (4)\mathtt{TAUT} \tag{5}$$

$$\vdash (\alpha \vee \alpha_1) \to \Box(\alpha \vee \alpha_1) \qquad \mathtt{Per} \tag{6}$$

$$\vdash \mathsf{K}(\alpha \vee \alpha_1) \to \mathsf{K}\Box(\alpha \vee \alpha_1) \qquad (6)\mathsf{S5_K} \tag{7}$$

$$\vdash \mathsf{K}\Box(\alpha \vee \alpha_1) \to \Box\mathsf{K}(\alpha \vee \alpha_1) \qquad \mathtt{PR} \tag{8}$$

$$\vdash \mathsf{K}(\alpha \vee \alpha_1) \to \Box\mathsf{K}(\alpha \vee \alpha_1) \qquad (7)(8)\mathtt{TAUT} \tag{9}$$

$$\vdash \mathsf{K}(\alpha \vee \alpha_1) \to (\widehat{\mathsf{K}}\alpha \vee \mathsf{K}\alpha_1) \qquad \mathsf{S5_K} \tag{10}$$

$$\vdash \Box\mathsf{K}(\alpha \vee \alpha_1) \to \Box(\widehat{\mathsf{K}}\alpha \vee \mathsf{K}\alpha_1) \qquad (10)\mathsf{S4_\Box} \tag{11}$$

$$\vdash \mathsf{K}(\alpha \vee \alpha_1) \to \Box(\widehat{\mathsf{K}}\alpha \vee \mathsf{K}\alpha_1) \qquad (9)(11)\mathtt{TAUT} \tag{12}$$

$$\vdash \mathsf{K}(\alpha \vee \alpha_1) \to \Box\varphi \qquad (12)\mathsf{S4_\Box} \tag{13}$$

$$\vdash \mathsf{K}(\alpha \vee \alpha_i) \to \Box\varphi \ (i \in \{1, \ldots, n\}) \qquad (13) \tag{14}$$

$$\vdash \psi \to \Box\varphi \qquad (4)(14)\mathtt{TAUT} \tag{15}$$

$$\vdash \neg\alpha \wedge \bigwedge_{1 \le i \le k} \widehat{\mathsf{K}}(\neg\alpha \wedge \neg\alpha_i) \to \Diamond(\mathsf{K}\neg\alpha \wedge \bigwedge_{1 \le i \le k} \widehat{\mathsf{K}}\neg\alpha_i) \qquad \mathtt{EU}_k \tag{16}$$

$$\vdash \neg\Diamond(\mathsf{K}\neg\alpha \wedge \bigwedge_{1 \le i \le k} \widehat{\mathsf{K}}\neg\alpha_i) \to \neg(\neg\alpha \wedge \bigwedge_{1 \le i \le k} \widehat{\mathsf{K}}(\neg\alpha \wedge \neg\alpha_i)) \qquad (16)\mathtt{TAUT} \tag{17}$$

$$\vdash \Box(\widehat{\mathsf{K}}\alpha \vee \bigvee_{1 \le i \le k} \mathsf{K}\alpha_i) \to (\alpha \vee \bigvee_{1 \le i \le k} \mathsf{K}(\alpha \vee \alpha_i)) \qquad (17)\mathsf{S4_\Box} \tag{18}$$

$$\vdash \Box\varphi \to \psi \qquad (18) \tag{19}$$

$$\vdash \Box\varphi \leftrightarrow \psi \qquad (15)(19)\mathtt{MP} \ \blacksquare \tag{20}$$

References

[1] Heyting Arend, Sur la logique intuitionniste, Bull. Cl. Sci., Acad. R. Belg. 16 (1930) 957–963.
[2] Philippe Balbiani, Alexandru Baltag, Hans van Ditmarsch, Andreas Herzig, Tomohiro Hoshi, Tiago De Lima, 'Knowable' as 'known after an announcement', Rev. Symb. Log. 1 (3) (2008) 305–334.
[3] Alexandru Baltag, Lawrence S. Moss, Sławomir Solecki, The logic of public announcements, common knowledge, and private suspicions, in: Readings in Formal Epistemology, Springer, 2016, pp. 773–812.
[4] Ivano Ciardelli, A first-order inquisitive semantics, in: 17th Amsterdam Colloquium, in: Lecture Notes in Computer Science, vol. 6042, Springer, 2009, pp. 234–243.
[5] Ivano Ciardelli, Inquisitive semantics and intermediate logics, Master's thesis, University of Amsterdam, 2009.
[6] Ivano Ciardelli, Dependency as question entailment, in: Dependence Logic, Springer International Publishing, Cham, 2016, pp. 129–181.
[7] Ivano Ciardelli, Questions in logic, PhD thesis, University of Amsterdam, 2016.
[8] Ivano Ciardelli, Questions as information types, Synthese 195 (1) (2018) 321–365.
[9] Ivano Ciardelli, Inquisitive Logic, 2021, Book draft.
[10] Ivano Ciardelli, Fausto Barbero, Undefinability in inquisitive logic with tensor, in: Proceedings of LORI VII, in: Lecture Notes in Computer Science, vol. 11813, Springer, 2019, pp. 29–42.
[11] Ivano Ciardelli, Floris Roelofsen, Inquisitive logic, J. Philos. Log. 40 (1) (2011) 55–94.
[12] Ivano Ciardelli, Floris Roelofsen, Inquisitive dynamic epistemic logic, Synthese 192 (6) (2015) 1643–1687.
[13] Ivano Ciardelli, Jeroen Groenendijk, Floris Roelofsen, On the semantics and logic of declaratives and interrogatives, Synthese 192 (6) (2015) 1689–1728.

[14] Ivano Ciardelli, Rosalie Iemhoff, Fan Yang, Questions and dependency in intuitionistic logic, Notre Dame J. Form. Log. 61 (1) (2020) 75–115.
[15] Raul Fervari, Andreas Herzig, Yanjun Li, Yanjing Wang, Strategically knowing how, in: Proceedings of International Joint Conference on Artificial Intelligence, IJCAI 2017, 2017, pp. 1031–1038.
[16] Kit Fine, Truth-maker semantics for intuitionistic logic, J. Philos. Log. 43 (2–3) (2014) 549–577.
[17] J. Hintikka, Knowledge and Belief: An Introduction to the Logic of the Two Notions, Cornell University Press, Ithaca N.Y, 1962.
[18] Jaakko Hintikka, Intuitionistic logic as epistemic logic, Synthese 127 (1/2) (2001) 7–19.
[19] Wesley H. Holliday, Inquisitive intuitionistic logic, in: Advances in Modal Logic 2020, College Publications, 2020, pp. 329–348.
[20] Yanjun Li, Yanjing Wang, Planning-based knowing how: a unified approach, Artif. Intell. 296 (2021) 103487.
[21] Mo Liu, Anantha Padmanabha, Ramaswamy Ramanujam, Yanjing Wang, Are bundles good deals for FOML?, CoRR, arXiv:2202.01581 [abs], 2022.
[22] Yu.T. Medvedev, Finitive problems, Dokl. Akad. Nauk SSSR 142 (1962) 1015–1018.
[23] Sergey A. Melikhov, A Galois connection between classical and intuitionistic logics. II. Semantics, preprint, arXiv:1312.2575, 2018.
[24] John-Jules Charles Meyer, Wiebe van der Hoek, Epistemic logic for AI and computer science, in: Cambridge Tracts in Theoretical Computer Science, vol. 41, Cambridge University Press, 1995.
[25] Pierangelo Miglioli, Ugo Moscato, Mario Ornaghi, Silvia Quazza, Gabriele Usberti, Some results on intermediate constructive logics, Notre Dame J. Form. Log. 30 (4) (1989) 543–562.
[26] Rani Nelken, Chung Chieh Shan, A modal interpretation of the logic of interrogation, J. Log. Lang. Inf. 15 (3) (2006) 251–271.
[27] Anantha Padmanabha, R. Ramanujam, Yanjing Wang, Bundled fragments of first-order modal logic: (un)decidability, in: FSTTCS 2018, in: LIPIcs, vol. 122, 2018, pp. 43:1–43:20.
[28] Vít Puncochár, Weak negation in inquisitive semantics, J. Log. Lang. Inf. 24 (3) (2015) 323–355.
[29] Vít Puncochár, Igor Sedlár, Inquisitive propositional dynamic logic, J. Log. Lang. Inf. 30 (1) (2021) 91–116.
[30] Johan van Benthem, Implicit and explicit stances in logic, J. Philos. Log. 48 (2018) 571–601.
[31] Johan van Benthem, Ştefan Minică, Toward a dynamic logic of questions, J. Philos. Log. 41 (2012) 633–669.
[32] Thom van Gessel, Action models in inquisitive logic, Synthese 197 (9) (2020) 3905–3945.
[33] Haoyu Wang, Yanjing Wang, Yunsong Wang, An epistemic interpretation of tensor disjunction, in: Advances in Modal Logic, AiML 2022, 2022, in press.
[34] Yanjing Wang, A new modal framework for epistemic logic, in: Proceedings of Conference on Theoretical Aspects of Rationality and Knowledge, TARK 2017, vol. 251, 2017, pp. 515–534.
[35] Yanjing Wang, A logic of goal-directed knowing how, Synthese 195 (10) (2018) 4419–4439.
[36] Yanjing Wang, Beyond knowing that: a new generation of epistemic logics, in: Outstanding Contributions to Logic, vol. 12, Springer Nature, 2018, pp. 499–533.
[37] Yanjing Wang, Knowing how to understand intuitionistic logic, manuscript, 2021.
[38] Yanjing Wang, Qinxiang Cao, On axiomatizations of public announcement logic, Synthese 190 (2013) 103–134.
[39] Chao Xu, Yanjing Wang, Thomas Studer, A logic of knowing why, Synthese 198 (2021) 1259–1285.
[40] Fan Yang, On extensions and variants of dependence logic: a study of intuitionistic connectives in the team semantics setting, PhD thesis, University of Helsinki, 2014.