Topological Semantics and Evidential-based Epistemic Logic

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Overview

Outline

Topological Space

A topological space is a pair (X, τ), where X is a nonempty set and $\tau \subseteq \mathcal{P}(X)$ is a family of subsets of X such that

- $\emptyset \in \tau$ and $X \in \tau$,
- τ is closed under arbitrary unions,
- τ is closed under finite intersection.

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- τ is closed under finite intersection.

Elements of τ are called open sets.
Complements of open sets are called closed sets.
An open set containing $x \in X$ is called an open neighbourhood of x.
A set $A \subseteq X$ is called clopen if it is both closed and open.

Example: Topology

- $\{\emptyset, X\}$ is called the trivial topology on X.
- The power set $\mathcal{P}(X)$ of X constitutes a topology on X called the discrete topology.
- On $\mathbb{R}$, let $\mathcal{B} = \{(a, c) \mid a, c \in \mathbb{R} \text{ and } a < c\}$. Then, for $O \subseteq \mathbb{R}, O \in \tau$ iff there exists some indexing set I such that $O = \bigcup_{i \in I} b_i$ where all $b_i \in \mathcal{B}$.
  τ is called the standard or natural topology on $\mathbb{R}$.

Interior points and Interior

Given a topological space (X, τ) :
An interior point of a set $A \subseteq X$ is a point $x \in X$ s.t. there exists an open neighborhood U of x with $U \subseteq A$. The interior of A is the set of all its interior points:

$$Int(A) = \{x \in X \mid \exists U \in \tau (x \in U \subseteq A)\}.$$

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$$Int(A) = \{x \in X \mid \exists U \in \tau (x \in U \subseteq A)\}.$$

It is easy to see that $Int(A)$ is the largest open subset of A, because $Int(A) = \bigcup\{U \in \tau \mid U \subseteq A\}$.

Limit points and Closure

A limit point of a set A is a point $x \in X$ s.t. every neighborhood U of x contains a point $y \in A$ with $y \neq x$. The closure of A is the set of all its limit points:

$$Cl(A) = \{x \in X \mid \forall U \in \tau (x \in U \Rightarrow U \cap A \neq \emptyset)\}.$$

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It is easy to see that $Cl(A)$ is the smallest closed set containing A, because $Cl(A) = \bigcap \{C \in \bar{\tau} \mid A \subseteq C\}$.

Epistemic interpretation

An *open set* U can be viewed as *a piece of evidence* that (imperfectly) indicates the true state of the world: the points in U are precisely those that are compatible with the evidence.

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Open sets as verifiable properties: Read $Int(A)$ as "A is known (or knowable)" based on evidence.

Closed sets as falsifiable property: Read $Cl(A)$ as "A is epistemically possible" (compatible with all evidence).

Some topological property

- A is dense if every open set $U \in \tau$ intersects A, i.e., if for all $U \in \tau, U \neq \emptyset \Rightarrow U \cap A \neq \emptyset$. Equivalently: A is dense iff $Cl(A) = X$.

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- A is nowhere dense if $Int(Cl(A)) = \emptyset$. Equivalently: if the interior of its complement $Int(X \setminus A)$ is dense. (In some papers, "almost all" is taken to mean "all points of the space except for a nowhere dense set")

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- A is nowhere dense if $Int(Cl(A)) = \emptyset$. Equivalently: if the interior of its complement $Int(X \setminus A)$ is dense. (In some papers, "almost all" is taken to mean "all points of the space except for a nowhere dense set")
- A topological space (X, τ) is connected if the only clopen sets are $\emptyset$ and X.
- A topological space (X, τ) is compact if every open cover of X has a finite subcover.

Basis

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A basis (or base) $\mathcal{B}$ is a subbasis satisfying in addition:

$$\forall B, B' \in \mathcal{B} \forall x \in B \cap B' \exists B'' \in \mathcal{B} \text{ s.t. } x \in B'' \subseteq B \cap B'$$

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Given a subbasis $\Sigma \subseteq \mathcal{P}(X)$, the topology τ_Σ generated by Σ on X is the smallest topology (on X) that includes Σ.

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Note: Not every Basis is closed under finite intersections.

Example: Subbasis and Basis

Example

- *The set* $\{(-\infty, a) \mid a \in \mathbb{Q}\} \cup \{(b, \infty) \mid b \in \mathbb{Q}\}$ *is a subbasis for the standard topology on* $\mathbb{R}$*.*
- *For the standard topology on* $\mathbb{R}$, $\{(a, b) \mid a < b, a, b \in \mathbb{R}\}$ *is a basis.*
- *For the standard topology on* $\mathbb{R}$, $\{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$ *is also a basis (it is a countable basis).*
- *For the standard topology on* $\mathbb{R}^2$, $\{(x, r) = \{y \in \mathbb{R}^2 \mid d(x, y) < r\} \mid x \in \mathbb{R}^2, r > 0\}$ *is a basis.*

Epistemic interpretation

EPISTEMOLOGY	TOPOLOGY
Directly observable basic evidence	Subbasis (Σ)
Directly observable combined evidence	Basis ($\mathcal{B}$)
Verifiable evidence	Open Sets (τ)
Factive evidence at x	Open neighbourhood $U \ni x$

Alexandroff Topology

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Let (X, R) be a preordered set. Then, the set $\tau_R = \{A \mid A \text{ is a upward closed set of } (X, R)\}$ is a topological space. We call τ_R is the upset topology.

Fact: every upset topology is an Alexandroff topology.

Alexandroff Topology

Give a topological space (X, τ) and two points $x, y \in X$, we say that x is a specialization of y, $x \sqsubseteq_\tau y$, if every (open) neighborhood of x is also a neighborhood of y:

$$x \sqsubseteq_\tau y \text{ iff } \forall U \in \tau(x \in U \Rightarrow y \in U)$$

Fact: Every open set is upwards-closed wrt the specialization preorder.

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Fact: Every open set is upwards-closed wrt the specialization preorder.

(X, τ) is an Alexandroff space iff $\tau = \tau_{\sqsubseteq_\tau}$.

Separation Axioms

The separation axioms are about the use of topological means to distinguish distinct points and disjoint sets.

T_0	$\forall x, y \in X, x \neq y \Rightarrow \exists U \in \tau((x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U))$
T_1	$\forall x, y \in X, x \neq y \Rightarrow \exists U \in \tau(x \in U \wedge y \notin U)$
T_2	$\forall x, y \in X, x \neq y \Rightarrow \exists U, V \in \tau(x \in U \wedge y \in V \wedge U \cap V = \emptyset)$
Regular	$\forall x \in X \forall A \in \bar{\tau}, x \notin A \Rightarrow \exists U, V \in \tau(x \in U \wedge A \subseteq V \wedge U \cap V = \emptyset)$
Normal	$\forall A, B \in \bar{\tau}, A \cap B = \emptyset \Rightarrow \exists U, V \in \tau(A \subseteq U \wedge B \subseteq V \wedge U \cap V = \emptyset)$

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T_1	$\forall x, y \in X, x \neq y \Rightarrow \exists U \in \tau(x \in U \wedge y \notin U)$
T_2	$\forall x, y \in X, x \neq y \Rightarrow \exists U, V \in \tau(x \in U \wedge y \in V \wedge U \cap V = \emptyset)$
Regular	$\forall x \in X \forall A \in \bar{\tau}, x \notin A \Rightarrow \exists U, V \in \tau(x \in U \wedge A \subseteq V \wedge U \cap V = \emptyset)$
Normal	$\forall A, B \in \bar{\tau}, A \cap B = \emptyset \Rightarrow \exists U, V \in \tau(A \subseteq U \wedge B \subseteq V \wedge U \cap V = \emptyset)$

$T_3 = T_0 +$ Regular, $T_4 = T_1 +$ Normal.

Metrizable Space

A topological space (X, τ) is said to be metrizable if there is a metric $d : X \times X \to [0, \infty)$ such that the topology induced by d is τ.

Note: Metrizable Space is different from Measurable Space.

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Theorem (Urysohn's metrization theorem)

If a topological space is T_3 and second-countable (has a countable base) then it is metrizable.

Kuratowski's Axioms

Let X be a set and $Int : \mathcal{P}(X) \to \mathcal{P}(X)$ an operator satisfying the following (Kuratowski) properties:

$$
\begin{aligned}
Int(X) &= X \\
Int(A) &\subseteq A \text{ for all } A \subseteq X \\
Int(A \cap B) &= Int(A) \cap Int(B) \text{ for all } A, B \subseteq X \\
Int(Int(A)) &= Int(A) \text{ for all } A \subseteq X
\end{aligned}
$$

Then $(X, \{A \in \mathcal{P}(X) \mid A = Int(A)\})$ forms a topological space.

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Then $(X, \{A \in \mathcal{P}(X) \mid A = Int(A)\})$ forms a topological space.A "Kuratowski" interior operator is an alternative to the standard definition of topology.

Outline

Modal Logic

Topological semantics of modal logic was introduced and developed by McKinsey and Tarski in 1930's and 1940's of the 20th century.

- A. Tarski, Der Aussagenkalkül und die Topologie, Fundam. Math. 31 (1938), 103-134.
- J. C. C. McKinsey, A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology, Journal of Symbolic Logic, vol. 6 (1941), pp. 117-134.
- J. C. C. McKinsey and A. Tarski, The algebra of topology, Annals of Mathematics, vol. 45 (1944), pp. 141-191.

One of the early reference along McKinsey and Tarski is Tang Tsao Chen (1938).

- T. Tsao-Chen, Algebraic postulates and a geometric interpretation for the Lewis calculus of strict implication, Bulletin of the American Mathematical Society, vol. 44 (1938), pp. 737-744. (National Wuhan University).

Tang Tsao Chen

Model Language

Let $\mathbf{P}$ denote the set of propositional letters.
The language of basic modal logic is defined by the grammar

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box\varphi$$

where $p \in \mathbf{P}$.
For other connectives, we assume the standard abbreviations.

Topological semantics

A topological model $\mathcal{M} = (X, \tau, \nu)$ is a tuple where (X, τ) is a topological space and ν a valuation, i.e., a map $\nu : \text{Prop} \to \mathcal{P}(X)$.

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The semantics for modal formulas is defined by the following inductive definition, where p is a propositional variable:

$$
\begin{array}{ll}
[\![\bot]\!] = \emptyset, & [\![p]\!] = \nu(p) \\
[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] & [\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!] \\
[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!] & [\![\Box\varphi]\!] = Int([\![\varphi]\!])
\end{array}
$$

Since $\Diamond\varphi = \neg\Box\neg\varphi$, we have $[\![\Diamond\varphi]\!] = Cl([\![\varphi]\!])$.

Topo-bisimulation

Definition

A topo-bisimulation between two topo-models $\mathcal{M} = (X, \tau, \nu)$ and $\mathcal{M}' = (X', \tau', \nu')$ is a non-empty relation $T \subseteq X \times X'$ such that if xTx' then:

- $x \in \nu(p) \Leftrightarrow x' \in \nu'(p)$ *for each* $p \in Prop$.
- *(forth):* $x \in U \in \tau \Rightarrow \exists U' \in \tau'$ *such that* $x' \in U'$ *and* $\forall y' \in U'\ \exists y \in U$ *such that* yTy'.
- *(back):* : $x' \in U' \in \tau' \Rightarrow \exists U \in \tau$ *such that* $x \in U$ *and* $\forall y \in U\ \exists y' \in U'$ *such that* yTy'.

As in the relational case if two points are linked by a topo-bisimulation, they are called topo-bisimilar.

Topo-bisimulation

Theorem

Let $\mathcal{M} = (X, \tau, \nu)$ and $\mathcal{M}' = (X', \tau', \nu')$ be two topo-models. Let $x \in X$ and $x' \in X'$ be topo-bisimilar points. Then for each modal formula φ we have

$$\mathcal{M}, x = \varphi \textit{ iff } \mathcal{M}', x' = \varphi$$

That is, modal formulas are invariant under topo-bisimulations.

Kuratowski's Axioms and S4

Kuratowski's axioms closely resemble the axioms of S4:

$Int(X) = X$	$\Box\top \leftrightarrow \top$
$Int(A) \subseteq A$	$\Box p \rightarrow p$
$Int(A \cap B) = Int(A) \cap Int(B)$	$\Box(p \land q) \leftrightarrow \Box p \land \Box q$
$Int(Int(A)) = Int(A)$	$\Box p \rightarrow \Box\Box p$

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$Int(A) \subseteq A$	$\Box p \to p$
$Int(A \cap B) = Int(A) \cap Int(B)$	$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$
$Int(Int(A)) = Int(A)$	$\Box p \to \Box\Box p$

This entails soundness of S4 for topological spaces.

Link to Kripke Semantics

When the underlying topology is Alexandroff given by the upsets wrt to a given preorder R on X, that is, τ_R, our topological semantics coincides with the standard Kripke semantics.

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When the underlying topology is Alexandroff given by the upsets wrt to a given preorder R on X, that is, τ_R, our topological semantics coincides with the standard Kripke semantics.

This entails completeness of S4 for all topological spaces.

Hence, S4 is sound and complete with respect to the class of all topological spaces (under the interior semantics).

Completeness proof via canonical model construction

The canonical topo-model $\mathcal{X}^c = (X^c, \tau^c, V^c)$ is defined as follows:

- $X^c = \{ T \subseteq \mathcal{L}_\Box : T$ maximally consistent $\}$,
- the canonical topology τ^c is generated by the "canonical basis"

$$\mathcal{B}^c = \{\widehat{\Box\varphi} : \varphi \in \mathcal{L}_\Box\},$$

  where $\widehat{\theta} = \{ T \in X^c : \theta \in T\}$, and
- the canonical valuation given by $V^c(p) = \widehat{p}$.

It is easy to see that $\mathcal{B}^c$ is indeed a basis (in fact, is closed under finite intersections and contains X^c).

McKinsey-Tarski Theorem

A topological space X is called dense-in-itself if X has no isolated points, i.e., there is no point $x \in X$ such that $\{x\}$ is open.

Theorem (McKinsey-Tarski, 1944)

S4 is the logic of an arbitrary (nonempty) dense-in-itself metric space.

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Remark

The original McKinsey-Tarski result had an additional assumption that the space is separable. In their 1963 book Rasiowa and Sikorski showed that this additional condition can be dropped. Their proof uses the Axiom of Choice.

Topological completeness

Theorem (McKinsey and Tarski, 1944)

- *S4 is complete wrt all topological spaces.*
- *S4 is complete wrt any dense-in-itself metrizable space X.*
- *S4 is complete wrt the real line $\mathbb{R}$.*
- *S4 is complete wrt the rational line $\mathbb{Q}$.*

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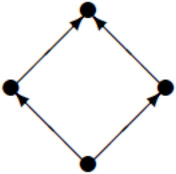
- S4.2 is sound and complete wrt the class of extremally disconnected spaces, in which the closure of every open subset is open.
- S4.3 is sound and complete wrt the class of hereditarily extremally disconnected topological spaces, in which every subspace is extremally disconnected.

Generalization of McKinsey-Tarski Theorem

Theorem (Bezhanishvili et al 2020)

There exists a normal space Z whose logic is the logic of the diamond iff there exists a measurable cardinal.



Topo-definability

Theorem

- *Neither compactness nor connectedness is topo-definable.*
- *None of the separation axioms is topo-definable.*

Outline

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What is Belief?

Most (but not all) philosophers accept that fully rational belief is

- consistent (though not necessarily factive),
- closed under entailment, and
- (unlike knowledge) fully introspective (both positively and introspectively).

This is because belief is purely subjective, thus (supposedly) totally transparent to the subject.

In other words, the logic of belief is commonly taken to be the modal logic $KD45_B$.

Stalnaker's logic for knowledge and belief

Stalnaker has proposed a logic intended to capture the relationship between knowledge and belief, where belief is interpreted in the strong sense of subjective certainty.

$$(\mathcal{L}_{KB}) \quad \varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid K\varphi \mid B\varphi$$

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This logic extends the classic $S4_K$ system for knowledge with the following additional axioms:

(D_B)	$B\varphi \to \neg B\neg\varphi$	Consistency of belief
(sPI)	$B\varphi \to KB\varphi$	Strong positive introspection
(sNI)	$\neg B\varphi \to K\neg B\varphi$	Strong negative introspection
(KB)	$K\varphi \to B\varphi$	Knowledge implies belief
(FB)	$B\varphi \to BK\varphi$	Full belief

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In this system, one can prove the following striking equivalence:

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where $\widehat{K}$ abbreviates $\neg K \neg$.

- Belief is equivalent to "the epistemic possibility of knowledge".
- In particular, belief can be defined in terms of knowledge-once you have knowledge, you get belief for free.

Stalnaker's logic for knowledge and belief

- (Full) belief can be defined in terms of knowledge as

$$B\varphi \leftrightarrow \widehat{K}K\varphi$$

- $\mathsf{KD45}_B$ as the logic of belief
- $\mathsf{S4.2}_K$ as the logic of knowledge

$$\mathsf{S4.2}_K = \mathsf{S4}_K + (\widehat{K}K\varphi \to K\widehat{K}\varphi)$$

The Topology of Full Belief[1]

Given a topo-model (X, τ, V), we interpret knowledge and belief as below:

$$[\![K\varphi]\!] = Int([\![\varphi]\!]), [\![B\varphi]\!] = Cl(Int([\![\varphi]\!]))$$

[1] Alexandru Baltag, Nick Bezhanishvili, Aybuke Ozgun and Sonja Smets, The Topology of Belief, Belief Revision and Defeasible Knowledge, LORI 2013

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Moreover, if we want to consider Dynamic revision, we need to restrict to hereditarily extremally disconnected spaces! In this case, every subspace of it is also extremally disconnected. Then the logic of knowledge will be S4.3.

[1] Alexandru Baltag, Nick Bezhanishvili, Aybuke Ozgun and Sonja Smets, The Topology of Belief, Belief Revision and Defeasible Knowledge, LORI 2013

The Topology of Weak Belief[2]

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Now, we need not to restrict to extremally disconnected spaces. Hence, the logic of knowledge is S4, and the logic of belief becomes wKD45.

$$\text{wKD45} = \mathbf{K} + (B\varphi \to \widehat{B}\varphi) + (B\varphi \to BB\varphi) + (B\widehat{B}B\varphi \to B\varphi)$$

[2] Alexandru Baltag, Nick Bezhanishvili, Aybuke Ozgun and Sonja Smets, The Topology of Full and Weak Belief, TbiLLC 2015

The Topology of Weak Belief[2]

Given a topo-model (X, τ, V), we interpret knowledge and belief as below:

$$[\![K\varphi]\!] = Int([\![\varphi]\!]), [\![B\varphi]\!] = Int(Cl(Int([\![\varphi]\!])))$$

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$$\text{wKD45} = \mathbf{K} + (B\varphi \to \widehat{B}\varphi) + (B\varphi \to BB\varphi) + (B\widehat{B}B\varphi \to B\varphi)$$

In fact, when K is S4.2, we have $B\varphi \leftrightarrow \widehat{K}K\varphi \leftrightarrow K\widehat{K}K\varphi$.

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Outline

Motivation

We are interested in studying notions of belief and knowledge, for a rational agent who is in possession of some (possibly false, possibly mutually contradictory) pieces of evidence.

This kind of topological semantics is heavily inspired by van Benthem & Pacuit[3].

[3] J. van Benthem and E. Pacuit, Dynamic Logics of Evidence-Based Beliefs. Studia Logica, 2011, 99(1): 61-92.

Evidence Models

Definition (van Benthem & Pacuit, 2011)

A (uniform) evidence model is a tuple $\mathcal{M} = (X, \mathcal{E}_0, V)$, where

- *$X \neq \emptyset$ is the set of possible worlds (or "states");*
- *$\mathcal{E}_0 \subseteq \mathcal{P}(X)$ is the set of basic evidence sets (also called "pieces of evidence"), satisfying $X \in \mathcal{E}_0$ and $\emptyset \notin \mathcal{E}_0$;*
- *$V : \mathbf{P} \to \mathcal{P}(X)$, where $\mathbf{P}$ is a set of propositional variables.*

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But evidence pieces are fallible (could be false), and could be mutually inconsistent.
An evidence e *is factive (or "correct") at world* x *if* $x \in e$*.*

Forming Beliefs based on (Fallible) Evidence

The main idea behind van Benthem & Pacuit's semantics:
The rational agent tries to form consistent beliefs, by looking at all maximally finitely-consistent "blocks" of evidence, and believing whatever is entailed by all of them.

- "Having evidence for φ need not imply belief."
- When forming beliefs, the agent should take all their available evidence for and against φ into account."

A (combined) evidence is any nonempty intersection of finitely many pieces of evidence. $\mathcal{E}$ is the family of all (combined) evidence:

$$\mathcal{E} := \{\bigcap F \mid F \in \mathcal{F}^{\text{fin}}\}$$

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- A (combined) evidence e supports P (or e is "evidence for" P) iff $e \subseteq P$.
- strength order $\supseteq$ on $\mathcal{E}$:

$$e \supseteq e' := e' \text{ is at least as strong as } e$$

The evidential plausibility order $\sqsubseteq_{\mathcal{E}}$ associated to an evidence model is defined by:

$$\begin{aligned} x \sqsubseteq_{\mathcal{E}} y &\text{ iff } \forall e \in \mathcal{E}_0 (x \in e \Rightarrow y \in e) \\ &\text{ iff } \forall e \in \mathcal{E} (x \in e \Rightarrow y \in e) \end{aligned}$$

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We denote the strict order by

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The set of "most plausible worlds" (maximal worlds wrt $\sqsubseteq_{\mathcal{E}}$) :

$$\mathrm{Max}_{\sqsubseteq_{\mathcal{E}}} X := \{y \in X \mid \forall z \in X (y \not\sqsubset_{\mathcal{E}} z)\}$$

A body of evidence is a family $F \subseteq \mathcal{E}_0$ of evidence pieces s.t. every finitely many of them are mutually consistent (finite intersection property):

$$\left(\forall F' \subseteq_{\text{in}} F\right)\left(F' \neq \emptyset \Rightarrow \bigcap F' \neq \emptyset\right)$$

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- A body of evidence F supports P iff $\bigcap F \subseteq P$.
- "strongest bodies of evidence":

$$\text{Max}_{\subseteq}(\mathcal{F}) := \left\{F \in \mathcal{F} \mid \forall F' \in \mathcal{F}\left(F \subseteq F' \Rightarrow F = F'\right)\right\}$$

Observation: $\text{Max}_{\subseteq}(\mathcal{F}) \neq \emptyset$ (Zorn's Lemma)

The Logic of Evidence, Belief and Infallible Knowledge

$$\mathcal{L}_0 := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid E_0\varphi \mid B\varphi \mid [\forall]\varphi$$

$E_0\varphi :=$ the agent has a basic (piece of) evidence for φ.
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$$\begin{array}{lll}
\mathcal{M}, x \models [\forall]\varphi & \text{iff} & [\![\varphi]\!]^{\mathcal{M}} = X \\
\mathcal{M}, x \models E_0\varphi & \text{iff} & \exists e \in \mathcal{E}_0 \left(e \subseteq [\![\varphi]\!]^{\mathcal{M}}\right) \\
\mathcal{M}, x \models B\varphi & \text{iff} & (\forall F \in Max_{\subseteq}(\mathcal{F})) \left(\bigcap F \subseteq [\![\varphi]\!]^{\mathcal{M}}\right) \\
 & \text{i.e.,} & \mathrm{Max}_{\sqsubseteq \mathcal{E}} X \subseteq [\![\varphi]\!]^{\mathcal{M}}
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So a proposition is believed (in the sense of van Benthem & Pacuit) iff it is supported by all the strongest bodies of evidence, or equivalently iff it is true in all the most plausible worlds.

Example

- Alice (a), a biology student, investigates an animal (unknown to her). She receives “pieces of evidence” from 4 different sources of information (her colleagues):

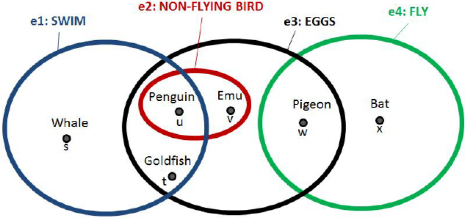
Source 1: it can swim (e_1)

Source 2: non-flying bird (e_2)

Source 3: it lays eggs (e_3)

Source 4: it flies (e_4)

Example



- **Worlds** $X = \{Whale, Penguin, Emu, Goldfish, Pigeon, Bat\}$
- **Evidence pieces** $\mathcal{E} = \{e_1, e_2, e_3, e_4, X\}$
- **Bodies of evidence**:

$\mathcal{F} = \{\{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_1, e_2\}, \{e_2, e_3\}$
$\{e_1, e_3\}, \{e_3, e_4\}, \{e_1, e_2, e_3\}, \{X\}, \{e_1, X\}, \{e_2, X\}, \{e_3, X\},$
$\{e_4, X\}, \{e_1, e_2, X\}, \{e_1, e_3, X\}, \{e_3, e_4, X\}, \{e_1, e_2, e_3, X\}\}$

- **Strongest bodies**: $Max_{\subseteq}(\mathcal{F}) = \{\{e_1, e_2, e_3, X\}, \{e_3, e_4, X\}\}$.
- **Beliefs**: $B(Penguin \vee Pigeon)$, $B(EGGS)$ (i.e. Be_3).
- **Non-beliefs**: $\neg B(e_1)$, $\neg B(e_2)$, $\neg B(e_4)$.

Consistency of Beliefs?

As we saw, a rational agent may receive mutually inconsistent pieces of evidence.
But shouldn't their rational beliefs still be consistent?

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- BUT: $B\bot$ can hold in some "bad" infinite models.

Solution: Instead of focusing on all the "strongest" such bodies, we may instead weaken the definition by looking at all finite bodies of evidence that are "strong enough".

Evidential Topology

The family of (combined) evidence $\mathcal{E}$ forms a topological basis.

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A topo-e-model is a tuple $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, where

- $(X, \mathcal{E}_0, V)$ is an evidence model,
- $\tau = \tau_{\mathcal{E}}$ is the (evidential) topology generated by $\mathcal{E}$.

Argument

An argument for P is a disjunction $U = \bigcup_{i \in I} e_i$ of evidences $e_i \in \mathcal{E}$, each separately supporting P (i.e. $e_i \subseteq P$ for all $i \in I$).

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- Epistemologically, an argument provides multiple evidential paths to support a common conclusion P.
- Topologically: a set of worlds $U \subseteq X$ is an argument (for something) iff it is open in the evidential topology (i.e. $U \in \tau_{\mathcal{E}}$.

Justification

A justification for P is an argument U for P that is consistent with every available evidence (i.e. $U \in \tau_{\mathcal{E}}$ such that $U \subseteq P$ and $U \cap e \neq \emptyset$ for all $e \in \mathcal{E}$).

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An argument (or justification) U is correct at x iff $x \in U$.

Characterizations of Belief

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$\exists U \in \tau(U \subseteq P \wedge Cl(U) = X)$.
- P includes a dense open set;

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So this definition really gives us a concept of justified belief: belief for which there exists some evidential justification.

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Topologically natural:
P is believed iff it's true in "almost all" worlds: i.e. all except for a nowhere-dense set.

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Topologically natural:

P is believed iff it's true in "almost all" worlds: i.e. all except for a nowhere-dense set.

Logically well-behaved:

This belief is always consistent (i.e. $B\bot$ never holds, since $Cl(Int(\emptyset)) = \emptyset$).

Overview

EPISTEMOLOGY	TOPOLOGY
Basic Evidence	Subbasis $(\mathcal{E}_0)$
(Combined) Evidence	Basis $(\mathcal{E})$
Arguments	Open Sets $(\tau_{\mathcal{E}_0})$
Justifications	Dense Open Sets
Justified Belief	Dense Interior
The weakest argument for P	$Int(P)$
Having true evidence for P	$x \in Int(P)$
Conditional Belief	"Conditional" Dense Interior
Infallible Knowledge	Global truth
Fallible Knowledge (of P)	$x \in \text{Int}(P)$ which is dense

The full Language $\mathcal{L}$

$$\mathcal{L} := p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid E_0\varphi \mid E\varphi \mid \Box_0\varphi \mid \Box\varphi \mid B\varphi \mid B^{\varphi}\varphi \mid K\varphi \mid [\forall]\varphi$$

$E_0\varphi :=$ the agent has a basic (piece of) evidence for φ
$E\varphi :=$ the agent has a (combined) evidence for φ
$\Box_0\varphi :=$ the agent has a factive basic (piece of) evidence for φ
$\Box\varphi :=$ the agent has factive (combined) evidence for φ

$B\varphi :=$ the agent has justified belief in φ
$B^{\varphi}\psi :=$ the agent believes that ψ conditionally on φ

$[\forall]\varphi :=$ the agent infallibly knows that φ
$K\varphi :=$ the agent fallibly (or defeasibly) knows that φ

The factive evidence fragment $\mathcal{L}_{[\forall]\Box_0\Box}$ having only $[\forall]$, $\Box_0$, and $\Box$ as its modalities can express all the other operators.

Axiomatization of the logic of factive evidence

the S5 axioms and rules for $[\forall]$
the S4 axioms and rules for $\Box$
$\Box_0\varphi \to \Box_0\Box_0\varphi$
$[\forall]\varphi \to \Box_0\varphi$
$\Box_0\varphi \to \Box\varphi$
$(\Box_0\varphi \wedge [\forall]\psi) \to \Box_0(\varphi \wedge [\forall]\psi)$
from $\varphi \to \psi$, infer $\Box_0\varphi \to \Box_0\psi$

Theorem

The logic of factive evidence has the finite model property, is decidable, and is completely axiomatized by the above system (wrt to topo-e-models).

Evidential Dynamics

"Hard" Updates: Move from an evidence model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ to the subspace model $\mathfrak{M}^{!\varphi} = (\|\varphi\|^{\mathfrak{M}}, \mathcal{E}_0^{!\varphi}, \tau^{!\varphi}, V^{!\varphi})$, where

$$\mathcal{E}_0^{!\varphi} = \{e \cap [\![\varphi]\!]^{\mathfrak{M}} : e \in \mathcal{E}_0 \text{ s.t. } e \cap [\![\varphi]\!]^{\mathfrak{M}} \neq \emptyset\}, \quad V^{!\varphi}(p) = V(p) \cap [\![\varphi]\!]^{\mathfrak{M}},$$

and

$$\tau^{!\varphi} = \{U \cap [\![\varphi]\!]^{\mathfrak{M}} : U \in \tau\}$$

is the topology generated by $\mathcal{E}_0^{!\varphi}$.

Evidential Dynamics

Evidence Addition: Move from the space $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ to the space $\mathfrak{M}^{+\varphi} = (X, \mathcal{E}_0 \cup \{[\![\varphi]\!]^{\mathfrak{M}}\}, \tau^{+\varphi}, V)$, where

$$\tau^{+\varphi} = \{V \cup (U \cap [\![\varphi]\!]^{\mathfrak{M}}) : V, U \in \tau\}$$

is the topology generated by $\mathcal{E}_0 \cup \{[\![\varphi]\!]^{\mathfrak{M}}\}$.

Outline

Subset Space Logics

Subset Space Logic (SSL) is a single-agent formalism for the notions of knowledge $K\varphi$ and effort $\Box\varphi$, where effort refers to any type of evidence-gathering, via, e.g., measurement, computation, approximation, experiment or announcement that can lead to an increase in knowledge.
(Moss and Parikh, 1992; Georgatos, 1993, 1994; Dabrowski et al., 1996).

Intersection Spaces

An intersection space is a pair $(X, \mathcal{O})$, where:

- X is a non-empty set of possible worlds;
- $\mathcal{O} \subseteq \mathcal{P}(X)$ ('observables' or 'evidence'), assumed to be closed under finite intersections.

Epistemically, $\mathcal{O}$ is the set of potential evidence, e.g. all possible results of measurements.
Our closure condition says that the (implicit) learner can cumulate observations (after observing two pieces of evidence U_1, U_2, her information state is given by $U_1 \cap U_2$) and that the tautological evidence X is always available (since $X = \bigcap \emptyset \in \mathcal{O}$).

SSL(Moss and Parikh(1992))

$$(\mathcal{L}_{K\Box})\, \varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi$$

$K\varphi :=$ the agent infallibly knows φ
$\Box\varphi := \varphi$ is stably true (under any further evidence-gathering)

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Given an intersection space model $\mathcal{X} = (X, \mathcal{O}, V)$ and an epistemic scenario (x, U) of $\mathcal{X}$,

$$\begin{array}{lll}
(x,U) \models p & \text{iff} & x \in V(p) \\
(x,U) \models \neg\varphi & \text{iff} & (x,U) \not\models \varphi \\
(x,U) \models \varphi \wedge \psi & \text{iff} & (x,U) = \varphi \text{ and } (x,U) \models \psi \\
(x,U) \models K\varphi & \text{iff} & (\forall y \in U)((y,U) \models \varphi) \\
(x,U) \models \Box\varphi & \text{iff} & \forall O \in \mathcal{O}(x \in O \subseteq U \Rightarrow (x,O) \models \varphi)
\end{array}$$

Multi-agent generalization

Target: generalize the topological arbitrary announcement setting to a multi-agent setting:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid \mathrm{int}(\varphi) \mid [\varphi]\varphi \mid \Box\varphi$$

- $K_i\varphi$: agent i knows φ.
- $\mathrm{int}(\varphi)$: 'φ is true and can be announced'.
- $[\varphi]\psi$: 'after announcement of φ, ψ (is true)' (Bjorndahl-style)
- $\Box\varphi$: corresponding arbitrary announcement modality (it is not the effort modality).

Main challenge

Straightforward Way

For each agent i, there is an open set U_i which represent the epistemic scenario of agent i.

For the case of two agents, instead of (x, U), the semantic primitive becomes a triple (x, U_i, U_j).

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For each agent i, there is an open set U_i which represent the epistemic scenario of agent i.

For the case of two agents, instead of (x, U), the semantic primitive becomes a triple (x, U_i, U_j).
$(x, U_i, U_j) \models K_i K_j p \iff$ for any $y \in U_i, (y, U_i, U_j) \models K_j p$
However, y may not be in U_j, (y, U_i, U_j) is not well-defined.

Multi-agent topological model

Given a topological space (X, τ), a neighbourhood function set Φ on (X, τ) is a set of (partial) neighbourhood functions $\theta : X \rightharpoonup (\mathcal{A} \rightarrow \tau)$:

- $x \in \theta(x)(i)$
- $\theta(x)(i) \subseteq \mathcal{D}(\theta)$
- $\forall y \in X$, $y \in \theta(x)(i)$ implies $y \in \mathcal{D}(\theta) \wedge \theta(x)(i) = \theta(y)(i)$
- $\theta|_U \in \Phi$

where $\mathcal{D}(\theta)$ is the domain of θ, and $\theta|_U$ is the neighbourhood function with $\mathcal{D}\left(\theta|_U\right) = \mathcal{D}(\theta) \cap U$ and $\theta|_U(x)(i) = \theta(x)(i) \cap U$.

Property of θ

1) θ is a partition for every agent i;
2) $\mathcal{D}(\theta)$ is open;

Multi-agent topologcial model

Definition

A multi-agent topological model (topo-model) is a tuple $\mathcal{M} = (X, \tau, \Phi, V)$, where (X, τ) is a topological space, Φ a neighbourhood function set, and V : **Prop** $\rightarrow \mathcal{P}(X)$ *a valuation function. The tuple $\mathcal{X} = (X, \tau, \Phi)$ is a multi-agent topological frame (topo-frame).*

A pair (x, θ) is called a neighbourhood situation if $x \in \mathcal{D}(\theta)$. The open set $\theta(x)(i)$ is called an epistemic neighbourhood at x of agent i.

Semantics

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid \mathrm{int}(\varphi) \mid [\varphi]\varphi \mid \Box\varphi$$

Given a topo-model $\mathcal{M} = (X, \tau, \Phi, V)$ and a neighbourhood situation $(x, \theta) \in \mathcal{M}$:

$\mathcal{M}, (x,\theta) \models K_i\varphi$	iff	$(\forall y \in \theta(x)(i))(\mathcal{M}, (y,\theta) \models \varphi)$
$\mathcal{M}, (x,\theta) \models Int(\varphi)$	iff	$x \in Int([\![\varphi]\!]^\theta)$
$\mathcal{M}, (x,\theta) \models [\varphi]\psi$	iff	$\mathcal{M}, (x,\theta) \models \mathrm{int}(\varphi)$ implies $\mathcal{M}, (x, \theta^\varphi) \models \psi$
$\mathcal{M}, (x,\theta) \models \Box\varphi$	iff	$(\forall \psi \in \mathcal{L}_{PAL_{int}})(\mathcal{M}, (x,\theta) \models [\psi]\varphi)$

where $\theta^\varphi = \theta|_{Int[\![\varphi]\!]^\theta}$ is an updated neighbourhood function.

Property of the settings

- $x \in \theta(x)(i)$: $\emptyset$ cannot be an epistemic neighbourhood
- $\theta(x)(i) \subseteq \mathcal{D}(\theta)$: $\forall y \in \theta(x)(i)$, (y, θ) will be well-defined
- $\forall y \in X, y \in \theta(x)(i) \implies \theta(x)(i) = \theta(y)(i)$: θ is a partition for agent i, hence K_i is S5
- $\theta|_U \in \Phi$: updated neighbourhood functions exist in Φ

Axiomatization

$S5_{K_i} + S4_{int} + (K_i\varphi \to \mathrm{int}(\varphi))$

([] – K)	$[\varphi](\chi \to \psi) \to ([\varphi]\chi \to [\varphi]\psi)$
(R1)	$[\varphi]p \leftrightarrow (\mathrm{int}(\varphi) \to p)$
(R2)	$[\varphi]\neg\psi \leftrightarrow (\mathrm{int}(\varphi) \to \neg[\varphi]\psi)$
(R3)	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
(R4)	$[\varphi]\,\mathrm{int}(\psi) \leftrightarrow (\mathrm{int}(\varphi) \to \mathrm{int}([\varphi]\psi))$
(R5)	$[\varphi]K_i\psi \leftrightarrow (\mathrm{int}(\varphi) \to K_i[\varphi]\psi)$
(R6)	$[\varphi][\psi]\chi \leftrightarrow [\neg[\varphi]\neg\,\mathrm{int}(\psi)]\chi$
(R7)	$\Box\varphi \to [\chi]\varphi$ where $\chi \in \mathcal{L}_{PAL_{int}}$
(DR1)	From φ and $\varphi \to \psi$, infer ψ
(DR2)	From φ, infer $K_i\varphi$
(DR3)	From φ, infer $\mathrm{int}(\varphi)$
(DR4)	From φ, infer $[\psi]\varphi$
(DR5)	From $\xi([\psi]\chi)$ for all $\psi \in \mathcal{L}_{PAL_{int}}$, infer $\xi(\Box\chi)$

Completeness

Theorem

$APAL_{int}$, PAL_{int} and EL_{int} are all sound and complete with respect to the class of all topo-models.

Proof Sketch

- Define the set of all MCS: X^c
- Define an equivalence relation $\sim_i$ on X^c: $\Gamma \sim_i \Delta$ iff $\forall\varphi(K_i\varphi \in \Gamma \text{ iff } K_i\varphi \in \Delta)$.
- The topology is generated by the subbasis

$$\Sigma = \{[\Gamma]_i \cap \widehat{\mathsf{int}(\varphi)}\}$$

Weak Multi-agent topological model

Definition

A weak multi-agent topological model (weak topo-model) is a topo-model $\mathcal{M} = (X, \tau, \Phi, V)$ as in S5-case with condition 3 replaced by
$\forall y \in X$, $y \in \theta(x)(i)$ implies $y \in \mathcal{D}(\theta)$ and $\theta(y)(i) \subseteq \theta(x)(i)$.

Theorem

The axiomatization of $wAPAL_{int}$, $wPAL_{int}$ and wEL_{int} are the corresponding system minus the 5 axiom.