A note on equivalence of two semantics for epistemic logic of shallow depths

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Abstract

This short note establishes the equivalence of the indexed semantics of [Wan06] and the epistemic world semantics of [KS03] for epistemic logic of bounded depths.

1   Introduction

This short note is dedicated to Prof. Frank Veltman who co-supervised my master thesis [Wan06], with Dr. Maricarmen Martinez, on a non-standard semantics for epistemic logic and its applications. The thesis was inspired, in spirit, by Frank's early work on update semantics [Vel96]. The basic ideas of the so-called *indexed semantics* proposed in the thesis are as follows:

- The meaning of (epistemic) modalities is *context-dependent*: more precisely, it depends on *where* the modality appears, e.g., the same 'Yanjing believes p' has different meanings in 'Yanjing believes Yanjing believes p' and in 'Frank believes Yanjing believes p', in the sense that Yanjing in Frank's mind may be quite different from Yanjing in his own mind.

- In the semantics, we have explicit epistemic relations for *imaginary agents*, e.g., not only epistemic relations for Yanjing and Frank but also Yanjing in Frank's mind, Frank in the mind of the imaginary Yanjing in Frank's mind and so on. In this way we can have a finer grip on the theory of mind and control the depth of epistemic reasoning.

Back then, I faced many troubles in axiomatizing logics based on such a semantics, but Frank generously gave a lot of support at the toughest moments, which I really appreciated. Although I was not very happy about the results in the thesis and the main content of the thesis was never published, the idea of the context/position-dependent semantics for modal logic shaped my latter research in a significant way, and it has become one of my favorite 'pet' tools. For example, the context-dependent treatment of modalities helped me to develop a dynamic epistemic framework in one chapter of my PhD dissertation [Wan10], where the meaning of actions are not attached to the actions themselves but due to protocols that agents are following (see also [Wan11b]). As another example, the context-dependent semantics gives me an alternative way of looking at dynamics in epistemic logic: the models can be kept the same but some simple syntactic contexts are recorded and changed. In this way, I can have a finer control over the dynamics, and it helps me to prove many incompleteness results for proof systems which were often taken for granted in the field of Dynamic Epistemic Logic (cf. [Wan11a, WC13]). This idea will appear in many other papers of mine in the future. I am really indebted to Frank and Maricarmen for their tolerance and encouragement to my 'weird' ideas: if I were 'advised' to do other more standard things, I might have done a better master thesis but would definitely miss one nice idea which can stay longer.

Actually, the idea is not that weird. Not only similar context-dependent semantics for modal logic appeared in other settings (e.g, [Gab02, BE09] which I only found out after my master),

but also an essentially equivalent semantics was developed by Kaneko and Suzuki to handle epistemic reasoning of shallow depths (in epistemic game theory)[KS03]. Their motivation is to bound the modality nesting in reasoning for a more realistic treatment of epistemic logic. The semantics is based on the *epistemic world models* which can be viewed as unravelings of the *indexed epistemic models* in my setting. In the next section, we will establish the equivalence of the two semantics in the sense that they validate the same set of (bounded) epistemic formulas. As a consequence, the complete proof system in [KS03] can be viewed as a complete proof system w.r.t. our semantics, which I regard as a humble present to Frank on this special occasion, since we struggled a lot to obtain the suitable completeness in my original setting.

2   The equivalence of the two semantics

Definition 1 (Epistemic Language **EL**). *Given a set $\boldsymbol{P}$ of propositional variables and a set $\boldsymbol{I}$ of agent names, the logical language $\boldsymbol{EL}$ is defined as:*

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i\varphi$$

where $p \in \boldsymbol{P}$ and $i \in \boldsymbol{I}$.

Let $\mathbf{I}^*$ be the collection of all the finite sequences of agent names in $\mathbf{I}$ and let $\mathbf{I}^+ = \mathbf{I}^*\backslash\{\epsilon\}$, i.e., the collection of the *non-empty* sequences. We define the *nesting set* of **EL** formulas $(n(\varphi) \subseteq \mathbf{I}^*)$ as follows:

$$\begin{array}{rclcrclcrcl}
n(\top) & = & \{\epsilon\} & & n(p) & = & \{\epsilon\} & & n(\neg\varphi) & = & n(\varphi) \\
n(\varphi \wedge \psi) & = & n(\varphi) \cup n(\psi) & & n(\Box_i\varphi) & = & \{i * s \mid s \in n(\varphi)\}
\end{array}$$

For example, $n(\Box_1(p \wedge \Box_2 q)) = \{1, 12\}$. We call a set $\mathbf{E} \subseteq \mathbf{I}^*$ an *epistemic ability set* if it is closed under taking initial segments.[1] Note that the empty sequence ϵ is always in $\mathbf{E}$.

Let $\mathbf{EL_E}$ be the fragment of **EL** such that the nesting of the epistemic modalities is restricted to the ones in $\mathbf{E}$ (thus 'shallow depth'), i.e. $n(\varphi) \subseteq \mathbf{E}$. For example, if $\mathbf{E} = \{\epsilon, 1, 2, 12\}$ then $\Box_2(p \wedge \Box_1 q)$ is not in $\mathbf{EL_E}$. Clearly **EL** is just $\mathbf{EL_{I^*}}$.

In [Wan06], I focused on $\mathbf{EL_{I^*}}$ but the semantics works for $\mathbf{EL_E}$ as well. Let us first define the models with explicit relations for imaginary agents.

Definition 2 (Indexed Epistemic Model for $\mathbf{EL_E}$ [Wan06]). *An indexed epistemic model for $\boldsymbol{EL_E}$ is a tuple $\mathcal{M} = \langle W, \{\rightarrow_s \mid s \in \boldsymbol{E}\backslash\{\epsilon\}\}, V\rangle$ where*

- *W is a non-empty set of possible worlds,*

- *$\rightarrow_s$ is a binary relation over W for each $s \in \boldsymbol{E}\backslash\{\epsilon\}$,*

- *valuation V assigns to each $p \in \boldsymbol{P}$ a set of worlds $V(p) \subseteq W$.*

A pointed indexed epistemic model for $\boldsymbol{EL}$ w.r.t. $\boldsymbol{E}$ is an indexed epistemic model with a designated point $w \in W$.

For any $s = i_0 i_1 \ldots i_k \in \mathbf{E}$, an *s-path* in an indexed epistemic model $\mathcal{M}$ is $w_0 i_1 w_1 \ldots i_k w_k$ such that for any $j \in [1, k-1]$, $w_j \rightarrow_{i_1 i_2 \cdots i_{j+1}} w_{j+1}$. Note that an ϵ-path from w is w itself.

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[1] In [KS03] the authors call $\mathbf{E}$ an *epistemic structure*, but it might be confused with Kripke structures in the epistemic logic setting.

In this note, to facilitate the precise correspondence with [KS03], we consider the pointed indexed epistemic model $\mathcal{M}, w$ with the following *seriality* property (where $*$ is the concatenation operator, e.g., $\epsilon * i * j = ij$):

> For any $s * i \in \mathbf{E}$, any s-path from w can be extended to an $s * i$-path.

In the sequel, when mentioning pointed indexed epistemic models, we assume they satisfy the above seriality property.

Definition 3 (Indexed Semantics for **EL** and **EL$_\mathbf{E}$** [Wan06]). *We first define the semantics for the full **EL** on indexed epistemic models w.r.t. $\mathbf{I}^*$. Given a model $\mathcal{M} = \langle W, \{\rightarrow_s \mid s \in \mathbf{I}^+\}, V \rangle$, the semantics is defined w.r.t strings s in $\mathbf{I}^*$ as follows:*

$\mathcal{M}, w \vDash \varphi$	$\Leftrightarrow$	$\mathcal{M}, w \vDash_\epsilon \varphi$
$\mathcal{M}, w \vDash_s \top$	$\Leftrightarrow$	*always*
$\mathcal{M}, w \vDash_s p$	$\Leftrightarrow$	$p \in V(w)$
$\mathcal{M}, w \vDash_s \neg\varphi$	$\Leftrightarrow$	$\mathcal{M}, w \nvDash_s \varphi$
$\mathcal{M}, w \vDash_s \varphi \wedge \psi$	$\Leftrightarrow$	$\mathcal{M}, w \vDash_s \varphi$ *and* $\mathcal{M}, w \vDash_s \psi$
$\mathcal{M}, w \vDash_s \Box_i\varphi$	$\Leftrightarrow$	*for all v such that $w \rightarrow_{s*i} v : \mathcal{M}, v \vDash_{s*i} \varphi$*

When $\mathbf{E} \subset \mathbf{I}^*$, the semantics of **EL$_\mathbf{E}$** on $\mathbf{E}$-indexed models w.r.t. $s \in \mathbf{E}$ ($\mathcal{M}, w \vDash_s \chi$) is essentially the same, under a condition of χ: $\{s * t \mid t \in n(\chi)\} \subseteq \mathbf{E}$. Note that otherwise the clause of $\Box_i$ may not be well-defined, e.g., $\mathcal{M}, w \vDash_{12} \Box_1 p$ cannot be defined on $\mathbf{E}$-indexed models where $\mathbf{E} = \{\epsilon, 1, 2, 12\}$ since there is no $\rightarrow_{121}$ relation.

We say φ is $\mathbf{E}$-valid ($\vDash_\mathbf{E} \varphi$) if for all pointed model $\mathcal{M}, w$ w.r.t. $\mathbf{E}$ we have $\mathcal{M}, w \vDash \varphi$.

For example, $\mathcal{M}, w \vDash \Box_1(p \wedge \Box_2 \neg p)$ in the following model $\mathcal{M}$:

$$w \xrightarrow{\ 1\ } p \xrightarrow{\ 12\ } \neg p$$
$$\searrow^2$$
$$p$$
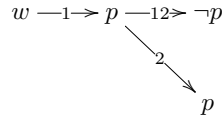
On the other hand, [KS03] proposed the following 'layered' model for **EL$_\mathbf{E}$**.

Definition 4 (Epistemic World Model for **EL$_\mathbf{E}$** [KS03]). *Given an epistemic ability set $\mathbf{E}$, an epistemic world model for **EL** w.r.t. $\mathbf{E}$ is a tuple $\mathcal{M} = \langle W, w_\epsilon, \{R_i \mid i \in \mathbf{I}\}, V \rangle$ where:*

- *W is a disjoint union of $\{W_s \mid s \in \mathbf{E}\}$ such that $W_\epsilon = \{w_\epsilon\}$,*

- *R_i is a binary relation over W such that for any $w \in W_s, v \in W_t$, if wR_iv then $t = s * i$,*

- *valuation V assigns to each $p \in \mathbf{P}$ a set of worlds $V(p) \subseteq W$,*

- *For any w, any $s * i \in \mathbf{E} : w \in W_s$ implies there exists $v \in W_{s*i}$ s.t. wR_iv.*

The last condition is clearly a version of seriality w.r.t. the epistemic ability.

Definition 5 (Epistemic World Semantics for **EL** and **EL$_\mathbf{E}$** [KS03]). *We first define the semantics for the full **EL** on epistemic world models w.r.t. $\mathbf{I}^*$. Given a model $\mathcal{N} = \langle W, \{R_i \mid i \in \mathbf{I}\}, V \rangle$, the semantics is defined as follows:*

$$\begin{array}{lll}
\mathcal{N}, w \Vdash \top & \Leftrightarrow & always \\
\mathcal{N}, w \Vdash p & \Leftrightarrow & p \in V(w) \\
\mathcal{N}, w \Vdash \neg\varphi & \Leftrightarrow & \mathcal{N}, w \nVdash \varphi \\
\mathcal{N}, w \Vdash \varphi \wedge \psi & \Leftrightarrow & \mathcal{N}, w \Vdash \varphi \text{ and } \mathcal{N}, w \Vdash \psi \\
\mathcal{N}, w \Vdash \Box_i\varphi & \Leftrightarrow & \text{for all } v \text{ such that } wR_iv : \mathcal{N}, v \Vdash \varphi
\end{array}$$

When $\boldsymbol{E} \subset \boldsymbol{I}^*$, the semantics of $\boldsymbol{EL_E}$ on $\boldsymbol{E}$-models ($\mathcal{N}, w \Vdash \chi$) is essentially the same, under the condition of χ: $w \in W_s$ implies $\{s * t \mid t \in n(\chi)\} \subseteq \boldsymbol{E}$. We say φ is $\boldsymbol{E}$-valid ($\Vdash_{\boldsymbol{E}} \varphi$) if for all epistemic world model $\mathcal{N}$ w.r.t. $\boldsymbol{E}$ we have $\mathcal{N}, w_\epsilon \Vdash \varphi$.

Note that the above semantics is essentially the standard Kripke semantics for modal logic but on special pointed Kripke models, where each world belongs to a unique 'layer' w.r.t. some $s \in \mathbf{E}$.

In the following, we show that the two semantics are equivalent.

Theorem 1. *Given any epistemic ability set $\boldsymbol{E}$:*

- *For each pointed indexed epistemic model $\mathcal{M}, w$ w.r.t. $\boldsymbol{E}$ there is an epistemic world model $\mathcal{N}$ w.r.t. $\boldsymbol{E}$ such that for any $\varphi \in \boldsymbol{EL_\epsilon}$: $\mathcal{M}, w \vDash \varphi \iff \mathcal{N}, w_\epsilon \Vdash \varphi$.*

- *For each epistemic world model $\mathcal{N}$ w.r.t. $\boldsymbol{E}$ there is a pointed indexed epistemic model $\mathcal{M}, w$ w.r.t. $\boldsymbol{E}$ such that for any $\varphi \in \boldsymbol{EL_E}$: $\mathcal{M}, w \vDash \varphi \iff \mathcal{N}, w_\epsilon \Vdash \varphi$.*

Proof. We define two truth preserving transformations between indexed epistemic models and epistemic world models in the following.

Given an indexed epistemic model $\mathcal{M} = \langle W, \{\to_s \mid s \in \mathbf{E} \backslash \{\epsilon\}\}, V \rangle$ and a $w \in W$, let $T(\mathcal{M}, w) = \langle W', w_\epsilon, \{R_i \mid i \in \mathbf{I}\}, V' \rangle$ where:

- W' is the union of $\{W_s \mid s \in \mathbf{E}\}$ such that $W_s = \{h \mid h \text{ is an } s\text{-path from } w \text{ in } \mathcal{M}\}$. Note that $w_\epsilon = w$.

- hR_ih' iff $h' = h * iv$ for some $v \in W$.

- $V'(h) = V(v)$ where $h = h' * v$ for some sequence h'.

Essentially, $T(\mathcal{M}, w)$ is an unraveling of $\mathcal{M}$ at w. Since $\mathcal{M}, w$ is serial it is clear that for any $h \in W_s$ if $s * i \in \mathbf{E}$ then there exists $h' \in W_{s*i}$ such that hR_ih'.

By induction on the structure of φ we can show that for any $s \in \mathbf{E}$, any $\mathbf{EL_E}$ formula φ such that $\{s * t \mid t \in n(\varphi)\} \subseteq \mathbf{E}$, any v in $\mathcal{M}$ such that there is an s-path h from w to v in $\mathcal{M}$:

$$\mathcal{M}, v \vDash_s \varphi \iff T(\mathcal{M}, w), h \Vdash \varphi.$$

The Boolean cases are trivial, we just show the case for $\varphi = \Box_i\psi$:

$$\begin{array}{ll}
& \mathcal{M}, v \vDash_s \Box_i\psi \\
\iff & \text{for all } u \text{ such that } v \to_{s*i} u : \mathcal{M}, u \vDash_{s*i} \psi \\
\iff & \text{for all } h' \text{ such that } hR_ih' : T(\mathcal{M}, w), h' \Vdash \psi \quad \text{(by def. of } R_i \text{ and induction hypothesis)} \\
\iff & T(\mathcal{M}, w), h \Vdash \Box_i\psi.
\end{array}$$

The induction hypothesis is that for *any* $s \in \mathbf{E}$, any s-path h from w to v the statement holds for any formula ψ which is less complex than φ such that $\{s * t \mid t \in n(\psi)\} \subseteq \mathbf{E}$. To apply the induction hypothesis, we need to check $\{s * i * t \mid t \in n(\psi)\} \subseteq \mathbf{E}$ which is guaranteed by the assumption $\{s * t \mid t \in n(\Box_i\psi)\} \subseteq \mathbf{E}$.

Now let $s = \epsilon$ then we have for all $\mathbf{EL_E}$ formula φ:

$$\mathcal{M}, w \vDash \varphi \iff T(\mathcal{M}, w), w_\epsilon \Vdash \varphi.$$

On the other hand, given an epistemic world model $\mathcal{N} = \langle W, w_\epsilon, \{R_i \mid i \in \mathbf{I}\}, V \rangle$, let $T'(\mathcal{N})$ be the indexed epistemic model $\langle W', \{\to_s \mid s \in \mathbf{E} \backslash \{\epsilon\}\}, V \rangle$ where:

- $W' = W$,

- $w \to_{s*i} v$ iff $w \in W_s, v \in W_{s*i}$ and wR_iv.

- $V'(w) = V(w)$.

Clearly, seriality is also preserved under the transaction T'.

Again by induction on the structure of φ we can show that any $s \in \mathbf{E}$, any $\mathbf{EL_E}$ formula φ such that $\{s * t \mid t \in n(\varphi)\} \subseteq \mathbf{E}$, any $v \in W_s$:

$$\mathcal{N}, v \Vdash \varphi \iff T'(\mathcal{N}), v \vDash_s \varphi.$$

Let $s = \epsilon$, we have:

$$\mathcal{N}, w_\epsilon \Vdash \varphi \iff T'(\mathcal{N}), w \vDash \varphi.$$

$\square$

The following corollary is immediate:

Corollary 2. *For any epistemic ability set $\mathbf{E}$, for any $\mathbf{EL_E}$ formula φ: $\vDash_{\mathbf{E}} \varphi \iff \Vdash_{\mathbf{E}} \varphi$.*

[KS03] introduces thought formulas $\square_s[\varphi]$ to capture the reasoning in the imaginary agent s's mind and obtain a complete proof system $\mathbf{GL_E}$. Due to the above corollary, $\mathbf{GL_E}$ is also complete w.r.t. $\vDash$ over the serial indexed epistemic models w.r.t. $\mathbf{E}$.

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