Epistmeic Logic XIII

Bundled fragments of first-order modal logic

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Nov. 30th, 2022

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Recap

A Logic of Mention-Some (and All)

Recap

The logic tool for knowing-wh

knowledge-that	—	propositional modal logic
knowledge-wh	—	first-order modal logic

In *Meaning and Necessity* (1947), Carnap remarked:

> *Any system of modal logic without quantification is of interest only as a basis for a wider system including quantification. If such a wider system were found to be impossible, logicians would probably abandon modal logic entirely.*

However, it seems that history went exactly the other way around.

Many things can be done in first-order modal logic

First-order modal logic is infamous for:

- issues in the semantics
- *quantifying-in* and substitution
- ambiguity: *de re* vs. *de dicto*
- incompleteness
- lack of Craig's interpolation
- undecidablility (hard to find useful decidable fragments)
- ....

At the same time, propositional modal logic is too successful...

Beyond knowing that: starting point

As philosophical logicians, we design specific-purpose languages to stay at the appropriate abstraction level to highlight the concepts in concern.

Instead of using the full language of first-order modal logic, we can use some well-behaved *fragments* of it to focus on what we really care but no more.

Can we repeat the success of propositional modal logic by a systematic approach to know-wh?

- simple language
- intuitive semantics
- useful models
- balanced expressive power and complexity...

The minimalist's "bundle" approach [Wang18]

- take a know-wh construction as a single modality (a "bundle"), e.g., pack $\exists x \Box (Mary \approx x)$ into *Kwho Mary*
- the use of quantifiers is restricted (recall the secret of success of propositional modal logic).
- natural and succinct to express the desired properties, e.g., I *know that* you *know what* the password is but I do not know the password.
- capture the essence of the relevant reasoning by axioms.
- lead to new decidable fragments of first-order modal logic.
- lead to intuitive understanding of non-classical logics.
- stay (technically) neutral for certain philosophical issues.

For each know-wh: the general steps

- focus on some (logically) interesting interpretation
- give natural semantics guided by the first-order modal formulation and linguistic/philosophical theories;
- axiomatize logics with (combinations of) new operators;
- simplify the semantics while keeping the validities;
- capture the expressivity via notions of bisimulation;
- dynamify those logics with new updates of knowledge;
- automate the inferences based on decidability;
- come back to philosophy and linguistics with new insights and questions.

Beyond knowing that: (technical) difficulties

- (apparently) not normal:
  - $\nvdash Kw(p \to q) \land Kw\ p \to Kw\ q$
  - $\nvdash Kh\varphi \land Kh\psi \to Kh(\varphi \land \psi)$
  - $\vdash \varphi \not\Rightarrow \vdash Ky\varphi$
- not strictly weaker: $\vdash Kw\varphi \leftrightarrow Kw\neg\varphi$;
- combinations of quantifiers and modalities, e.g., $\exists x \Box \varphi(x)$;
- the things that we quantify vary a lot;
- the axioms depend on the special shape of φ as well;
- weak language vs. rich model: hard to axiomatize;
- fragments of FO/SO-modal language: we know little.

Characteristic feature

How to distinguish the work in this line and other related work in the literature?

Whether it uses a single modality for a type of know-wh, instead of breaking it down into quantifiers, normal modalities, questions, predicates and so on.

It also gives us a new “looking glass” to understand the world.

Some knowing-wh logics we proposed and studied

wh-word	bundle	connection	key ref
whether	$\Box\varphi \vee \Box\neg\varphi$	non-contingency logic	[FWvD14,15
what	$\exists x \Box(\varphi \to x \approx c)$	weakly aggregative logic	[WF13,14]
how	$\exists \pi \Box[\pi]\varphi$	game logic, ATL	[Wang15,17
why	$\exists t \Box(t : \varphi)$	justification logic	[XWS18]

We obtained complete axiomatizations, characterizations of expressive power, and decidability ...

Along the way, we understand better why neighbourhood-like semantics works for various philosophical logics.

Connections to existing logics and linguistic theories

Classification by question words:

- Knowing whether: non-contingency logic, ignorance logic
- Knowing what: weakly aggregative logic, dependence logic
- Knowing how: game Logic, alternating temporal logic
- Knowing why: (quantified) justification Logic
- Knowing who: (dynamic) termed modal logic

Classification by logical forms:

- *Mention-some*: e.g., *knowing how/why...* $\exists x \mathcal{K} \varphi(x)$
- *Mention-all* (strongly exhaustive reading): e.g., I *know who* came to the party... $\forall x(\mathcal{K}\varphi(x) \vee \mathcal{K}\neg\varphi(x))$
- In-between: *know-value* $\exists x(\mathcal{K}\ c \approx x) \leftrightarrow \forall x(\mathcal{K}\ c \approx x \vee \mathcal{K}\ c \not\approx x)$

Epistemic logic: form one to many

(Routine) research questions:

- Model theory, proof theory, computational complexity
- Group knowledge
- Logical omniscience
- Natural dynamics
- Applications

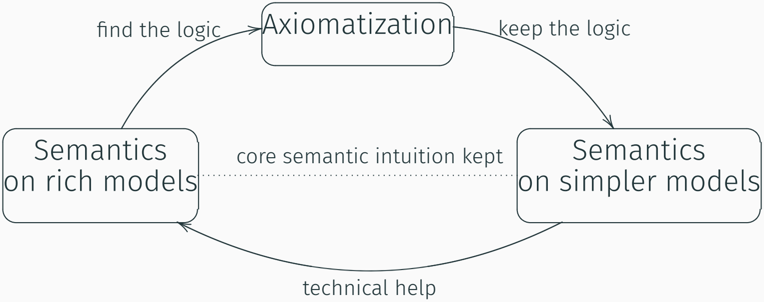
New questions:

- Interactions of different knowledge expressions;
- Simplification of semantics.

Simplify the semantics while keeping the logic

Common difficulties: weak language vs. rich semantics

To restore the balance between the language and model:



Disadvantages of those concrete logics

'Disadvantages' from a linguistic point of view:

- Compositionality
- Uniformity
- Expressivity

Disadvantages in terms of knowledge representation:

- Propositional epistemic logic is not really about the *content* of knowledge!

Towards a general new framework

What we are after:

- Expressive enough: covering the essence of those non-standard epistemic logics
- Not too much: sharing most good properties of propositional modal logic

Uniformity, compositionality, expressvity, computability: we want a predicate modal framework like the propositional modal logic

A new framework for predicate epistemic logic

Inspired by the concrete know-wh logics, we introduce the bundle modalities into the predicate modal language:

- pack $\exists x\mathcal{K}$ into a *bundle* modality (mention-some)
- pack $\forall x\Delta$ into a *bundle* modality (mention-all)

You can also come up with your favourite bundles inspired by the categorization of the penex forms for the *classical decision problem*.

We obtain some nice and powerful fragments of first-order modal logic.

A Logic of Mention-Some (and All)

A logic framework of mention-some [Wang TARK17]

Definition ($\exists\Box$-fragment)

Given set of variables X and set of predicate symbols Ps,

$$\varphi ::= P\overline{x} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x \Box \varphi$$

where $x, y \in \mathrm{X}$, $P \in \mathrm{Ps}$.

$\exists x \Box \varphi$ reads 'I know some x such that $\varphi(x)$'.

$\Box\varphi$ is expressible by $\exists x \Box \varphi$ if x does not occur free in φ.

We can add the equality symbol, function symbols, and constants.

Expressivity

- Knowing-wh: $\exists x \Box \varphi(x)$
- "I know a theorem of which I do not know any proof": $\exists x \Box \neg \exists y \Box Prove(y, x)$
- "a knows a country which b knows its capital": $\exists x \Box_a \exists y \Box_b Capital(y, x)$

First-order Kripke semantics

Definition (First-order Kripke Model)

An *increasing domain* model $\mathcal{M} = \langle W, D, \delta, R, \rho \rangle$ where:

- W is a non-empty set.
- D is a non-empty set.
- $R \in 2^{W \times W}$ is a binary relation over W.
- $\delta : W \to 2^D$ assigns to each $w \in W$ a *non-empty* local domain s.t. wRv implies $\delta(w) \subseteq \delta(v)$ for any $w, v \in W$.
- $\rho : \mathrm{Ps} \times W \to \bigcup_{n \in \omega} 2^{D^n}$ such that ρ assigns each n-ary predicate on each world an n-ary relation on D.

We write $D_w^{\mathcal{M}}$ for the local domain $\delta(w)$ in $\mathcal{M}$. If $\delta(w) = \delta(w')$ for all w, w' then it is called a *constant domain* model.

Semantics

Definition ($\exists\Box$ Semantics)

$$
\begin{aligned}
\mathcal{M}, w, \sigma \models \exists x \Box \varphi \quad &\Leftrightarrow \quad \text{there exists an } a \in D_w^{\mathcal{M}} \text{ such that} \\
&\quad\;\; \mathcal{M}, v, \sigma[x \mapsto a] \models \varphi \text{ for all } v \text{ s.t. } wRv \\
&\Leftrightarrow \quad \text{there exists an } a \in D_w^{\mathcal{M}} \text{ such that} \\
&\quad\;\; \mathcal{M}, w, \sigma[x \mapsto a] \models \Box \varphi
\end{aligned}
$$

$\exists\Box$ fragment is indeed an extension of ML:

$\models \Box\varphi \leftrightarrow \exists x \Box \varphi$ (given x does not appear free in φ).

A formula φ is *satisfiable* if there is an increasing domain pointed model $\mathcal{M}, w$ and an assignment σ such that $\mathcal{M}, w, \sigma \models \varphi$ and $\sigma(x) \in D_w^{\mathcal{M}}$ for all $x \in X$.

$\exists\Box$-Bisimulation (inspired by monotonic and obj-world bis)

Given $\mathcal{M}$ and $\mathcal{N}$, non-empty $Z \subseteq (W_\mathcal{M} \times D^*_\mathcal{M}) \times (W_\mathcal{N} \times D^*_\mathcal{N})$ is call an $\exists\Box$-*bisimulation*, if for every $((w, \bar{a}), (v, \bar{b})) \in Z$ such that $|\bar{a}| = |\bar{b}|$ the following holds (we write $w\bar{a}$ for $(w, \bar{a})$):

PISO $\bar{a}$ and $\bar{b}$ form a "partial isomorphism" based on the interpretations of predicates at w and v respectively.

$\exists\Box$Zig For any $c \in D^\mathcal{M}_w$, there is a $d \in D^\mathcal{N}_v$ such that for any $v' \in W_\mathcal{N}$ if vRv' then there exists w' in $W_\mathcal{M}$ such that wRw' and $w'\bar{a}cZv'\bar{b}d$. ($\forall^{object}_\mathcal{M}\exists^{object}_\mathcal{N}\forall^{world}_\mathcal{N}\exists^{world}_\mathcal{M}$)

$\exists\Box$Zag Symmetric to $\exists\Box$Zig.

We say $\mathcal{M}, w\bar{a}$ and $\mathcal{N}, v\bar{b}$ are $\exists\Box$-*bisimilar* ($\mathcal{M}, w\bar{a} \underline{\leftrightarrow}_{\exists\Box} \mathcal{N}, v\bar{b}$) if $|a| = |b|$ and there is an $\exists\Box$-bisimulation linking $w\bar{a}$ and $v\bar{b}$.
If there is equality symbol then PISO should respect *identity*.

Example

Consider the *constant domain* models $\mathcal{M}$ and $\mathcal{N}$:

$$\mathcal{M}: \quad \begin{tikzcd} \underline{w} \arrow[r] \arrow[dr] & v : Pa \\ & u : Pb \end{tikzcd} \qquad \mathcal{N}: \quad \begin{tikzcd} \underline{s} \arrow[r] \arrow[dr] & t : Pc \\ & r \end{tikzcd}$$

where $D^{\mathcal{M}} = \{a, b\}$, $D^{\mathcal{N}} = \{c\}$. Suppose P is the only unary predicate, we can show that $\mathcal{M}, w \underline{\leftrightarrow}_{\exists\Box} \mathcal{N}, s$ by an $\exists\Box$-bisimulation Z:

$$\{(w, s), (va, tc), (ub, tc), (vb, rc), (ua, rc)\}$$

Note that $\exists\Box$Zig and $\exists\Box$Zag hold trivially for $w\bar{a}$ and $v\bar{b}$ if w and v *do not* have any successor, based on the fact that local domains are non-empty by definition.

Limited expressive power

Theorem

$\mathcal{M}, w\bar{a} \leftrightarroweq_{\exists\Box} \mathcal{N}, v\bar{b}$ then $\mathcal{M}, w\bar{a} \equiv_{\mathsf{MLMS}^{\approx}} \mathcal{N}, v\bar{b}$.

Proposition

$\Box\exists xPx$, $\exists x\Diamond Px$ and $\Diamond\exists xPx$ are not *expressible in the $\exists\Box$-fragment.*

For the undefinability of $\Box\exists Px$ see the previous example.

For $\exists x\Diamond Px$, and $\Diamond\exists xPx$, consider:

$$\mathcal{M}: \quad \begin{tikzcd} \underline{w} \arrow[r] \arrow[dr] & v : Pa \\ & u \end{tikzcd} \qquad \mathcal{N}: \quad \begin{tikzcd} \underline{s} \arrow[r] & t \end{tikzcd}$$

where $D^{\mathcal{M}} = \{a, b\}$, $D^{\mathcal{N}} = \{c\}$ as before.

A model $\mathcal{M}$ is said to be $\exists\Box$-*saturated*, if for any $w \in W^{\mathcal{M}}$, and any finite sequence $\overline{a} \in D^*_{\mathcal{M}}$:

$\exists\Box$-type If for each finite subset Δ of a set $\Gamma(\overline{y}x)$ where $|\overline{y}| = |\overline{a}|$, $\mathcal{M}, w \vDash \exists x \Box \bigwedge \Delta[\overline{a}]$, then there is a $c \in D^{\mathcal{M}}_w$ such that $\mathcal{M}, w \vDash \Box\varphi[\overline{a}c]$ for all $\varphi \in \Gamma$, where x is assigned c.

$\Diamond$-type If for each finite subset Δ of $\Gamma(\overline{x})$ such that $|\overline{x}| = |\overline{a}|$, $\mathcal{M}, w \vDash \Diamond \bigwedge \Delta[\overline{a}]$, then there is a v such that wRv and $\mathcal{M}, v \vDash \varphi[\overline{a}]$ for each $\varphi \in \Gamma$.

Theorem

For $\exists\Box$-saturated models $\mathcal{M}, \mathcal{N}$ and $|\overline{a}| = |\overline{b}|$:
$\mathcal{M}, w\overline{a} \leftrightarrows_{\exists\Box} \mathcal{N}, v\overline{b} \Leftrightarrow \mathcal{M}, w\overline{a} \equiv_{\mathbf{MLMS}^{\approx}} \mathcal{N}, v\overline{b}$

Theorem (Wang TARK17)

A first-order modal formulas is equivalent to a formula in the $\exists\Box$-fragment iff it is invariant under $\exists\Box$-bisimulation.

A complete epistemic logic over S5 models SMLMS

Over S5 (constant-domain) models, **MLMS** is very powerful, it can also express *mention-all* by $\forall x \Diamond(\Box\varphi \vee \Box\neg\varphi)$ (also $\forall x \Box\varphi$ by $\forall x \Diamond\Box\varphi$).

Axioms

`TAUT`	all axioms of propositional logic
`DISTK`	$\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$
`T`	$\Box\varphi \to \varphi$
`4MS`	$\exists x\Box\varphi \to \Box\exists x\Box\varphi$
`5MS`	$\neg\exists x\Box\varphi \to \Box\neg\exists x\Box\varphi$
`KtoMS`	$\Box(\varphi[y/x]) \to \exists x\Box\varphi$ (admissible $\varphi[y/x]$)
`MStoK`	$\exists x\Box\varphi \to \Box\varphi$ (if $x \notin FV(\varphi)$)
`MStoMSK`	$\exists x\Box\varphi \to \exists x\Box\Box\varphi$
`KT`	$\Box\top$

Rules:

`MP`	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
`MONOMS`	$\dfrac{\vdash \varphi \to \psi}{\vdash \exists x\Box\varphi \to \exists x\Box\psi}$

To treat the equality (if we introduce it), we also need `ID` : $x \approx x$ and `SUBID` : $x \approx y \to (\varphi \to \psi)$. We can derive `KEQ` : $x \approx y \to \Box(x \approx y)$ and `KNEQ` : $x \not\approx y \to \Box(x \not\approx y)$.

We can axiomatize the logic over arbitrary models without `T`, `4MS`, `5MS`, `MStoMSK`.

Compare with the know-how logic

TAUT	all axioms of propositional logic	MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
DISTK	$\mathcal{K}p \wedge \mathcal{K}(p \to q) \to \mathcal{K}q$	NECK	$\dfrac{\varphi}{\mathcal{K}\varphi}$
T	$\mathcal{K}p \to p$	EQREPKh	$\dfrac{\varphi \to \psi}{\mathcal{K}h\varphi \to \mathcal{K}h\psi}$
4	$\mathcal{K}p \to \mathcal{K}\mathcal{K}p$	SUB	$\dfrac{\varphi(p)}{\varphi[\psi/p]}$
5	$\neg\mathcal{K}p \to \mathcal{K}\neg\mathcal{K}p$		
AxKtoKh	$\mathcal{K}p \to \mathcal{K}hp$		
AxKhtoKhK	$\mathcal{K}hp \to \mathcal{K}h\mathcal{K}p$		
AxKhtoKKh	$\mathcal{K}hp \to \mathcal{K}\mathcal{K}hp$		
AxKhKh	$\mathcal{K}h\mathcal{K}hp \to \mathcal{K}hp$		
AxKhbot	$\neg\mathcal{K}h\bot$		

Completeness proof

Definition

A set of **MLMS**$^+$ formulas has $\exists$-property if for each $\exists x\Box\varphi \in \mathbf{MLMS}^+$ it contains a "witness" formula $\exists x\Box\varphi \to \Box\varphi[y/x]$ for some $y \in \mathrm{X}^+$ where $\varphi[y/x]$ is admissible.

Definition (Canonical model)

The canonical model is a tuple $\langle W^c, D^c, \sim^c, \rho^c\rangle$ where:

- W^c is the set of maximal SMLMS$^+$-consistent sets with $\exists$-property,
- $D^c = \mathrm{X}^+$,
- $s \sim^c t$ iff $\Box(s) \subseteq t$ where $\Box(s) := \{\varphi \mid \Box\varphi \in s\}$,
- $\overline{x} \in \rho^c(P, s)$ iff $P\overline{x} \in s$.

It is routine to show that $\sim^c$ is an equivalence relation

Completeness proof

Lemma

If $\Box\psi \notin s \in W^c$ then there exists a $t \in W^c$ such that $s \sim^c t$ and $\neg\psi \in t$.

The witnesses for $\exists\Box$ formulas can be added by using:

$$\vdash_{\mathsf{SMLMS}} (\exists x \Box\varphi \to \Box\psi) \to \Box(\exists x \Box\varphi \to \Box\psi).$$

Lemma

Let σ^* be the assignment such that $\sigma^*(x) = x$ for all $x \in X^+$. For any $\varphi \in$ MLMS^+, any $s \in W^c$:

$$\mathcal{M}^c, s, \sigma^* \vDash \varphi \Leftrightarrow \varphi \in s$$

Each SMLMS consistent set can be extended to an SMLMS^+ consistent set.

Axiomatizations over other classes of frames [Xun Wang 21]

Axioms:

TAUT	all axioms of propositional logic
DISTK	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\Box$to$\exists\Box$	$\Box\varphi[y/x] \to \exists x\Box\varphi$ (if $\varphi[y/x]$ is admissible)

Rules:

MP $\dfrac{\varphi, \varphi \to \psi}{\psi}$ NEC $\dfrac{\varphi}{\Box\varphi}$ $\mathrm{R}^{\mathrm{i}}\Box\mathrm{to}\exists\Box$ $\dfrac{\Box\varphi \to \psi}{\exists x\Box\varphi \to \psi}(x \notin FV(\psi))$

Plus the corresponding axioms for frame conditions:

$$\textbf{D} \quad \Box\varphi \to \Diamond\varphi, \textbf{T} \quad \Box\varphi \to \varphi$$

$$\textbf{4} \quad \Box\varphi \to \Box\Box\varphi$$

$$\textbf{5} \quad \neg\Box\varphi \to \Box\neg\Box\varphi$$

What about decidability?

The situation for first-order modal logic is hopeless. Simply putting a decidable fragment of first-order logic plus a modality does not work at all.

Language	Decidability	Ref
P^1	undecidable	[Kripke 62]
x, y, p, P^1	undecidable	[Gabbay 93]
$x, y, \Box_i$, single P^1	undecidable	[Rybakov & Shkatov 17]

The decidable fragments are rare (only one x in $\Box$). Most of the propositional know-wh logics are in the one variable fragment.

Language	Decidability	Ref
single x	decidable	[Segerberg 73]
$x, y / P^1 /$ GF, $\Box_i(x)$	decidable	[Wolter & Zakharyaschev 01]

Tableaux (can be viewed as a satisfiability game)

We start from negated normal form (and require some "cleanness"):

$$\varphi ::= P\overline{x} \mid \neg P\overline{x} \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \exists x \Box \varphi \mid \forall x \Diamond \varphi$$

$\dfrac{w : \varphi_1 \vee \varphi_2, \Gamma, \sigma}{w : \varphi_1, \Gamma, \sigma \mid w : \varphi_2, \Gamma, \sigma}\ (\vee) \qquad \dfrac{w : \varphi_1 \wedge \varphi_2, \Gamma, \sigma}{w : \varphi_1, \varphi_2, \Gamma, \sigma}\ (\wedge)$
Given $n \geq 0, m \geq 1$: $\dfrac{w : \Box^{x_1}\varphi_1, \ldots, \Box^{x_n}\varphi_n, \Diamond^{y_1}\psi_1, \ldots, \Diamond^{y_m}\psi_m, l_1 \ldots l_k, \sigma}{\{(wv^{y}_{y_i} : \{\varphi_j \mid 1 \leq j \leq n\}, \psi_i[y/y_i], \sigma') \mid y \in Dom(\sigma'), i \in [1, m]\}}\ (\texttt{BR})$
Given $n \geq 1, k \geq 0$: $\dfrac{w : \Box^{x_1}\varphi_1, \ldots, \Box^{x_n}\varphi_n, l_1 \ldots l_k, \sigma}{w : l_1 \ldots l_k, \sigma}\ (\texttt{END})$

where $\sigma' = \sigma \cup \{(x_j, x_j) \mid j \in [1, n]\}$ and $l_k \in lit$ (the literals), $\Box^x$ is abbreviation of $\exists x \Box$, $\Diamond^x$ denotes $\forall x \Diamond$

An example

$$
\begin{array}{c}
w : \{\exists x \Box (Px \vee Qx) \wedge \forall y \Diamond \neg Qy \wedge \neg Pz, \{(z,z)\} \quad (\wedge) \times 2 \\
| \\
w : \{\exists x \Box (Px \vee Qx), \forall y \Diamond \neg Qy, \neg Pz\}\{(z,z)\} \quad (\mathtt{BR})
\end{array}
$$

$$
\begin{array}{ccc}
wv_y^x : \{Px \vee Qx, \neg Qx\}, \{(x,x),(z,z)\} & (\vee) & wv_y^z : \{Px \vee Qx, \neg Qz\}, \{(x,x),(z,z)\} \\
| & & | \\
wv_y^x : \{Px, \neg Qx\}, \{(x,x),(z,z)\} & & wv_y^z : \{Qx, \neg Qz\}, \{(x,x),(z,z)\}
\end{array}
$$

Theorem (Wang TARK17)

A formula φ in the $\exists\Box$ fragment is satisfiable is satisfiable iff its NNF has an open tableau.

Theorem (Wang TARK17)

A formula φ in the $\exists\Box$ fragment is satisfiable over arbitrary increasing domain models then it has an finite tree model whose depth is linearly bound by the length of φ.

Corollary (Wang TARK17)

Satisfiability checking of $\exists\Box$ fragment over arbitrary increasing domain is PSPACE-complete.

The $\exists\Box$ fragment behaves like the basic propositional modal logic but much more powerful.

Moreover, we can show that:

Theorem (Padmanabha, Ramanujam, Wang FSTTCS18)

The $\exists\Box$-fragment is decidable over arbitrary constant domain models.

Actually we can show that:

Theorem (Padmanabha, Ramanujam, Wang FSTTCS18)

The $\exists\Box$-fragment cannot distinguish increasing domain and constant domain models. The logic is exactly the same over constant domain models or increasing domain models.

Bad news

The meaning of the world is the separation of wish and fact.

— Gödel

- $\exists\Box$ fragment is undecidable over S5 models: replacing each quantifier in a first-order formula in the prenex form by $\exists x\Box$ or $\forall x\Diamond\Box$ respectively qua satisfiability
- $\forall\Box$ fragment with two unary predicates is undecidable over constant domain models: use $\Diamond\big(P(x) \wedge Q(y)\big)$ to encode the binary predicate, and use $\forall z_1\Box\, \forall z_2\Box\, \big(\Diamond^n\Diamond\, (P(z_1) \wedge Q(z_2)) \to \Box^n\Diamond(P(z_1) \wedge Q(z_2))\big)$ to force uniformity of evaluation.

It is not as robust as propositional modal logic: we are at the edge of first-order expressivity.

Joint work with Liu, Padmanabha, Ramanujam [MFCS 2022]

Over increasing domain models:

Domain	$\forall\Box$	$\exists\Box$	$\Box\forall$	$\Box\exists$	Upper/ Lower Bound
Increasing	✓	✗	✗	✗	PSpace-complete
	✗	✓	✗	✗	
	✗	✗	✓	✗	
	✗	✗	✗	✓	ExpSpace/ PSpace
	✓	✓	✗	✗	ExpSpace/NexpTime
	✗	✗	✓	✓	
	$\star$	✓	✓	$\star$	Undecidable
	✗	✓	✗	✓	No FMP
	✓	✓	✗	✓	Undecidable
	✓	✗	✓	✓	ExpSpace/ NexpTime
	loosely bundled				

Joint work with Liu, Padmanabha, Ramanujam [MFCS 2022]

We can also allow $\exists x \beta$ where β is a boolean combination of atomic formulas and modal formulas. Moreover, we can allow a quantifier alternation of the form $\exists x_1 \cdots \exists x_n \, \forall y_1 \cdots \forall y_m \; \beta$. As we will see, the fact that the existential quantifiers are outside the scope of universal quantifiers can help us to obtain decidability results over increasing domain models.

Definition (LBF syntax)

The loosely bundled fragment of $\mathsf{FOML}^{\approx}$ is the set of all formulas constructed by the following syntax of α:

$$\begin{aligned} \psi ::= &\quad P(z_1, \dots z_n) \mid \neg P(z_1, \dots z_n) \mid \psi \wedge \psi \mid \psi \vee \psi \mid \Box \alpha \mid \Diamond \alpha \\ \alpha ::= &\quad \psi \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \exists x_1 \dots \exists x_k \forall y_1 \dots \forall y_l \; \psi \end{aligned}$$

where $k, l, n \geq 0$ and $P \in \mathrm{Ps}$ has arity n and $x_1, \dots x_k$, $y_1, \dots y_l$, $z_1, \dots, z_n \in \mathrm{X}$.

Joint work with Liu, Padmanabha, Ramanujam [MFCS 2022]

Over constant domain models:

Domain	$\forall\Box$	$\exists\Box$	$\Box\forall$	$\Box\exists$	Upper/ Lower Bound
Constant	✓	$\star$	$\star$	$\star$	Undecidable
	$\star$	$\star$	✓	$\star$	
	✗	✓	✗	✗	PSPACE-complete
	✗	✗	✗	✓	No FMP
	✗	✓	✗	✓	

Further directions:

- What about adding $\approx$ and constant symbols (for decidability)?
- Which frame conditions can be added while keeping the decidability.
- Model/proof theoretical aspects.

See Xun Wang, Yuanzhe Yang's incoming work.

Logic of mention-all [Zhouhang Zhou 18]

Recall that Δ is the knowing whether operator.

Definition (The $\forall\Delta$-fragment)

Given X and Ps, the fragment $\forall\Delta$ is defined as:

$$\varphi ::= P\overline{x} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Delta\varphi \mid \forall x \Delta\varphi$$

The semantics for Δ is as usual.

Axioms

`TAUT`	all axioms of propositional logic
`KwCon`	$\Delta\varphi \wedge \Delta\psi \to \Delta(\varphi \wedge \psi)$
`KwDis`	$\Delta\varphi \to \Delta(\varphi \to \psi) \vee \Delta(\neg\varphi \to \chi)$
`KwTop`	$\Delta\top$
`KwEq`	$\Delta\varphi \leftrightarrow \Delta\neg\varphi$
`KtoMS`	$\forall x \Delta\varphi \to \Delta(\varphi[y/x])$ (if $\varphi[y/x]$ is admissible)
`WB`	$\forall x \Delta(\psi \to (\Delta\varphi \wedge \gamma)) \to \Delta(\psi \to (\forall x \Delta\varphi \wedge \gamma))$ (if x occurs free only in φ)

Rules:

`MP` $\dfrac{\varphi, \varphi \to \psi}{\psi}$ `REKw` $\dfrac{\vdash \varphi \leftrightarrow \psi}{\vdash \Delta\varphi \leftrightarrow \Delta\psi}$ `RKwtoMA` $\dfrac{\vdash \varphi \to \Delta\psi}{\vdash \varphi \to \forall x \Delta\psi}$ (

The system is sound and complete over constant-domain models.