Epistemic Logic (IX)

Logics of knowing whether and much more

Yanjing Wang
Department of Philosophy, Peking University
Nov. 15th, 2023

`wangyanjing.com`

Background

Understanding the expressivity

Axiomatizations

Background

> *All happy families are alike; each unhappy family is unhappy in its own way.*
>
> Leo Tolstoy, *Anna Karenina*

Whether the second half is possible depends on how many kinds of unhappiness there are (w.r.t. the number of families)...

(Un)happiness may depend on knowledge...

A loves B (p) but B doesn't love A (q);

p, q, $K_A q$;

p, q, $K_A q$, and $K_A K_B p$;

...

Sometimes, it is better not to know...

> *All omniscient agents are alike; each ignorant agent is ignorant in its own way.*

How many different ignorant states are there?

Given a single fact p in the 2-agent setting, how many different epistemic states (maximal consistent sets in $\mathbb{S}5$-**EL**) are there?

If the language is countable, there are at most $2^{\aleph_0}$ many consistent sets, but can we really realize $2^{\aleph_0}$ many mutually inconsistent consistent sets? Is it trivial? What about using only one proposition letter?

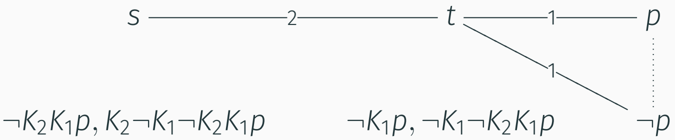
- $(p, K_1 p), (p, \neg K_1 p)$
- from the first: $(p, K_1 p, K_2 K_1 p), (p, K_1 p, \neg K_2 K_1 p)$
- from the second: $(p, \neg K_1 p, K_2 \neg K_1 p), (p, \neg K_1 p, \neg K_2 \neg K_1 p)$
- It seems we can continue like this arbitrarily...
- $(p, K_1 p, \neg K_2 K_1 p, \neg K_1 \neg K_2 K_1 p, K_2 \neg K_1 \neg K_2 K_1 p)$: inconsistent!
- In particular, $K_2 \neg K_1 \neg K_2 K_1 p$ and $\neg K_2 K_1 p$ are inconsistent.

Epistemic logic was invented to check consistency of knowledge and ignorance (Hintikka).

To show $\{K_2\neg K_1\neg K_2K_1p, \neg K_2K_1p\}$ is inconsistent:

(1)	$\neg K_1p \to \neg K_2K_1p$	$(\mathtt{T})$
(2)	$K_1\neg K_1p \to K_1\neg K_2K_1p$	$(\mathtt{NEC}, \mathtt{DIST})$
(3)	$\neg K_1p \to K_1\neg K_1p$	$(\mathtt{5})$
(4)	$\neg K_1p \to K_1\neg K_2K_1p$	$(\mathtt{MP}(2)(3))$
(5)	$\neg K_1\neg K_2K_1p \to K_1p$	
(6)	$K_2\neg K_1\neg K_2K_1p \to K_2K_1p$	$(\mathtt{NEC}, \mathtt{DIST})$

We can also show it semantically (recall completeness theorem: a consistent set of formulas must be satisfiable!)



The first answer was provided by Aumann (1989)

Aumann showed that there are $2^{\aleph_0}$ many mutually inconsistent consistent sets. Hart, Heifetz, and Samet (1996) presented a much simpler construction using "knowing whether": Define $Kw_i\phi = K_i\phi \vee K_i\neg\phi$. For any sequence $s \in \{0,1\}^+$ construct ϕ_s:

- $\phi_0 = \neg p$; $\phi_1 = p$
- if $|s| > 0$ is odd: $\phi_{sx} = \begin{cases} \neg Kw_1\phi_s & x = 0 \\ Kw_1\phi_s & x = 1 \end{cases}$
- if $|s| > 0$ is even: $\phi_{sx} = \begin{cases} \neg Kw_2\phi_s & x = 0 \\ Kw_2\phi_s & x = 1 \end{cases}$

E.g., $\phi_{0101} = Kw_1\neg Kw_2 Kw_1\neg p$, $\phi_{1010} = \neg Kw_1 Kw_2 \neg Kw_1 p$.

We can show $\Phi_w = \{\phi_s \mid s \text{ is an non-empty initial segment of } w\}$ are all consistent for each $w \in \{0,1\}^\omega$.

Let w_k be the kth position of w, e.g., $(01...)_1 = 0$, $(0010...)_3 = 1$.

Build a "canonical" model: $\mathcal{M} = \langle \Phi_w \mid w \in \{0,1\}^\omega, \sim_i, V \rangle$ where

- $\Phi_w \sim_1 \Phi_v$ iff $w_k = v_k$ for all the even k and if $w_k = v_k = 1$ for some even k then $w_{k-1} = v_{k-1}$.
- $\Phi_w \sim_2 \Phi_v$ iff $w_k = v_k$ for all the odd $k > 1$ and if $w_k = v_k = 1$ for some odd $k > 1$ then $w_{k-1} = v_{k-1}$.
- $V(p) = \{\Phi_w \mid w_0 = 1\}$.

We can show that $\Phi_w \models \phi_s$ if s is an initial segment of w by induction on the length of s.

Key observation: $\models Kw_i\phi \leftrightarrow Kw_i\neg\phi$ and the rule of replacement for equals for Kw_i: if $w = 0001...$ you want to show $\neg Kw_2\neg Kw_1\neg p$ holds on all the worlds $\Phi_v \sim_1 \Phi_w$ (equiv. $\neg Kw_2 Kw_1 p$, $\neg Kw_2 Kw_1 \neg p$).

Fragments of **EL** [Parikh and Krasucki 92]; more generally see Klein and Pacuit (manuscript, presented at LOFT).

"knowing whether" is useful

Natural and succinct to express:

- knowledge expression with hidden content: *A* "I *know whether p* but I won't tell you."
- knowledge with ignorance: *B* "I don't *know whether p* but I know you *know whether p*."

To be used (e.g., in muddy children):

- As assumptions: children *know whether* others are dirty.
- As precondition of actions: e.g., "step forward if you *know whether* you are dirty".
- As goals of planning: e.g., eventually the children *know whether* they are dirty.

Using *Kw* can give us (exponential) succinctness (try to unravel ϕ_S we had before).

A much more general setting

Essentially modal logic is about necessity $\Box$ (and possibility $\Diamond$) in various contexts:

- epistemic: know that
- doxastic: believe that
- deontic: ought to be that
- proof theoretical: it is provable that

General setting

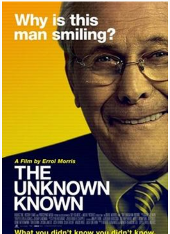
$$\Delta\phi := \Box\phi \vee \Box\neg\phi \qquad \nabla\phi := \neg\Delta\phi = \neg\Delta\neg\phi = \neg\Box\phi \wedge \neg\Box\neg\phi$$

Modal logic with Δ and ∇ as the primitive modality is about contingency and non-contingency in various contexts:

- alethic ∇: contingency [Montgomery & Routley 66 and many more, see our RSL article for a survey]
- epistemic ∇: ignorance [van der Hoek & Lomuscio 03]
- doxastic ∇: not opinionated about
- deontic ∇: moral indifference [von Wright 51]
- proof theoretical ∇: undecided [Zolin 2001]

The unknown unknown

> *Reports that say that something hasn't happened are always interesting to me, because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don't know we don't know. And if one looks throughout the history of our country and other free countries, it is the latter category that tend to be the difficult ones.*
>
> — *Donald Rumsfeld*



Connections with rough sets [Guan, Deng, Wang, Li 2021]

Given a set W with an equivalence relation R representing uncertainty of different objects. Given a set $X \subseteq W$:

- the upper approximation of X is $R^*(X) = \bigcup\{Y \mid Y \in W/_R, X \cap Y \neq \emptyset\}$, i.e., the things which could be in X.
- the lower approximation of X is $R_*(X) = \bigcup\{Y \mid Y \in W/_R, Y \subseteq X\}$, i.e., the things must in X.
- X is precise (not rough) with respect to R iff $R^*(X) = R_*(X)$. A rough set is the pair $\langle R^*(X), R_*(X)\rangle$.

$\mathcal{M}, w \vDash \Delta\phi \iff \mathcal{M}, w \vDash \Box\phi \vee \Box\neg\phi$.

Then $\mathcal{M} \vDash \Delta\phi$ iff $[\![\phi]\!]$ is precise iff $[\![\neg\phi]\!]$ is precise.

Understanding the expressivity

Non-contingency (knowing whether) operator

NCL is defined as follows:

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Delta_i \phi$$

where $p \in \mathbf{P}$ and $i \in \mathbf{I}$. A Kripke model $\mathcal{M}$ is a triple

$$\langle S, \{\rightarrow_i \mid i \in \mathbf{I}\}, V \rangle$$

where S is a non-empty set, $\rightarrow_i \subseteq S \times S$ and $V : \mathbf{P} \rightarrow 2^S$.

$$\begin{array}{lcl} \mathcal{M}, s \vDash \Delta_i \phi & \Leftrightarrow & \text{for all } t_1, t_2 \text{ such that } s \rightarrow_i t_1, s \rightarrow_i t_2 : \\ & & (\mathcal{M}, t_1 \vDash \phi \Leftrightarrow \mathcal{M}, t_2 \vDash \phi) \\ & \Leftrightarrow & \text{either for all } t \text{ such that } s \rightarrow_i t : \mathcal{M}, t \vDash \phi \\ & & \text{or for all } t \text{ such that } s \rightarrow_i t : \mathcal{M}, t \nvDash \phi \end{array}$$

NCL is clearly no more expressive than ML since we can define a translation t : NCL $\to$ ML such that:

$$t(\Delta_i \phi) \quad = \quad \Box_i t(\phi) \vee \Box_i \neg t(\phi)$$

What about the other way around? It is also easy if we restrict ourselves to reflexive models: we can define a translation t' : ML $\to$ NCL such that:

$$t'(\Box_i \phi) \quad = \quad t'(\phi) \wedge \Delta_i t'(\phi)$$

Are they equally expressive over arbitrary models? If not, how to characterize the expressive power of NCL within ML?

Standard bisimilarity is too strong for NCL

Definition (Standard Bisimulation)

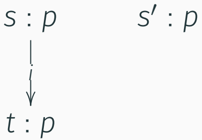
Let $\mathcal{M} = \langle S, \{\to_i \mid i \in \mathrm{I}\}, V\rangle$, $\mathcal{N} = \langle S', \{\to'_i \mid i \in \mathrm{I}\}, V'\rangle$ be two models. A binary relation Z over $S \times S'$ is a *bisimulation* between $\mathcal{M}$ and $\mathcal{N}$, if Z is non-empty and whenever sZs':

- (Invariance) s and s' satisfy the same propositional variables;
- (Zig) if $s \to_i t$, then there is a t' such that $s' \to_i t'$ and tZt';
- (Zag) if $s' \to_i t'$, then there is a t such that $s \to_i t$ and tZt'.

$\mathcal{M}, s$ is *bisimilar* to $\mathcal{N}, t$ ($\mathcal{M}, s \leftrightarroweq \mathcal{N}, t$) if there is a bisimulation between $\mathcal{M}$ and $\mathcal{N}$ linking s with t.

Bisimilarity is too strong for NCL

Standard bisimilarity is clearly an invariance relation for NCL but it is too strong:



These two finite models satisfy the same NCL formulas but they are clearly not bisimilar.

But in most of the cases when there are two and more successors the standard bisimulation seems fine.

A crucial observation

To tell the subtle difference we need to connect Δ and $\Box$:

$\Box_i$ is *almost definable* by Δ_i (call the schema AD):

Proposition

For any ϕ, ψ, $\vDash \neg\Delta_i\psi \to (\Box_i\phi \leftrightarrow (\Delta_i\phi \wedge \Delta_i(\psi \to \phi)))$.

This inspires us to:

- come up with a structural equivalence notion of Δ_i-*bisimulation* and characterize the expressive power;
- come up with the right definition of canonical relations in the latter completeness proofs;
- find the right axioms for special frame properties.

Δ-bisimulation

If there are two successors which can be told apart by **NCL** formulas then the bisimulation conditions should work. However, to make it purely *structural* is quite non-trivial. Idea: define it within a *single* model.

Definition (Δ-Bisimulation)

Let $\mathcal{M} = \langle S, R, V \rangle$ be a model. A binary relation Z over S is a Δ-*bisimulation* on $\mathcal{M}$, if Z is non-empty and whenever sZs':

- (Invariance) s and s' satisfy the same basic p;
- (Zig) if there are two different successors t_1, t_2 of s such that $(t_1, t_2) \notin Z$ and $s \to_i t$, then $\exists\, t'$ s.t. $s' \to_i t'$ and tZt';
- (Zag) if there are two different successors t'_1, t'_2 of s' such that $(t'_1, t'_2) \notin Z$ and $s' \to_i t'$, then $\exists\, t$ s.t. $s \to_i t$ and tZt'.

Δ-Bisimilarity

$\mathcal{M}, s$ and $\mathcal{N}, t$ are Δ-bisimilar ($\mathcal{M}, s \leftrightarrows_\Delta \mathcal{N}, t$) if there is a Δ-bisimulation on the *disjoint union* of $\mathcal{M}$ and $\mathcal{N}$ linking s and t.

Don't forget to show that Δ-bisimilarity is indeed an *equivalence relation*! Transitivity is quite hard!

Theorem (Fan, Wang, van Ditmarsch AiML14)

For image-finite (or **NCL** *saturated models)* $\mathcal{M}, s$ *and* $\mathcal{N}, t$*:* $\mathcal{M}, s \leftrightarrows_\Delta \mathcal{N}, t \iff \mathcal{M}, s \equiv_{\mathbf{NCL}} \mathcal{N}, t$ *(satisfying the same* **NCL** *formulas).*

Proof ideas: ($\Longrightarrow$) suppose $\mathcal{M}, s \nvDash \Delta_i\phi$ show that $\mathcal{N}, t \nvDash \Delta_i\phi$ by Zig and IH. ($\Longleftarrow$) take $\equiv_{\mathbf{NCL}}$ as Z and use the AD schema to express $\Diamond \bigwedge \Gamma$ for Zig.

The use of Δ-bisimilarity: model expressivity

Theorem (Fan, Wang, van Ditmarsch AiML14)

NCL is the Δ-bisimilarity invariant fragment of ML (and FOL).

Proof ideas:

- Since NCL and ML are compact, we can show NCL is the $\equiv_{\mathsf{NCL}}$-invariant fragment of ML.
- Replace $\equiv_{\mathsf{NCL}}$ by $\underline{\leftrightarrow}_{\Delta}$: show that each $\underline{\leftrightarrow}_{\Delta}$ invariant formula is also $\equiv_{\mathsf{NCL}}$ invariant, by using ultrafilter extension and saturation based on the Hennessy-Milner-like theorem.

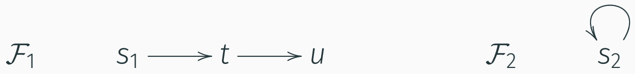
For characterizarion w.r.t. FOL we need to use van Benthem theorem and the fact that $\underline{\leftrightarrow}_{\Delta}$ is coarser than $\underline{\leftrightarrow}$.

Is the difference in expressivity just a negligible subtlety?

Theorem

*The frame properties of seriality, reflexivity, transitivity, symmetry, and Euclidicity are **not** definable in **NCL**.*

Proof idea:



We can show $\mathcal{F}_1 \vDash \phi \iff \mathcal{F}_2 \vDash \phi$ by its contrapositive and $\underline{\leftrightarrow}_\Delta$. Then towards contradiction...

Axiomatizations

Axiomatizations of NCL over various frame classes

Apparent difficulties:

- NCL formulas cannot capture the frame properties.
- NCL is not normal:
  - $\Delta_i(\phi \to \psi) \wedge \Delta_i\phi \to \Delta_i\psi$ is not valid.
  - $\Delta(\phi \wedge \psi) \to (\Delta\phi \wedge \Delta\psi)$ is not valid.
  - Monotonicity rule is not valid.
- With extra axioms like $\Delta_i\phi \leftrightarrow \Delta\neg\phi$.

Consider the following axiom schemas and rules as system $\mathbb{SNCL}$:

`TAUT`	all instances of tautologies
`KwCon`	$\Delta_i\phi \wedge \Delta_i\psi \to \Delta_i(\phi \wedge \psi)$
`KwDis`	$\Delta_i\phi \to \Delta_i(\phi \to \psi) \vee \Delta_i(\neg\phi \to \chi)$
`KwNeg`	$\Delta_i\phi \leftrightarrow \Delta_i\neg\phi$
`KwTop`	$\Delta_i\top$
`MP`	From ϕ and $\phi \to \psi$ infer ψ
`REKw`	From $\phi \leftrightarrow \psi$ infer $\Delta_i\phi \leftrightarrow \Delta_i\psi$

As a theorem: $\Delta(\phi \vee \psi) \wedge \Delta(\phi \to \psi) \to \Delta\psi$.

Theorem

$\mathbb{SNCL}$ is sound and strongly complete w.r.t. ***NCL*** *over the class of arbitrary frames (and serial frames).*

The proof is based on the following canonical model construction, inspired by the "almost definability" schema AD:

$$\neg\Delta_i\psi \to (\Box_i\phi \leftrightarrow (\Delta_i\phi \wedge \Delta_i(\psi \to \phi)))$$

Definition (Canonical model)

Define $\mathcal{M}^c = \langle S^c, R^c, V^c\rangle$ as follows:

- $S^c = \{s \mid s \text{ is a maximal consistent set of } \mathbb{SNCL}\}$
- For all $s, t \in S^c$, $sR^c_i t$ iff there exists χ such that:
  - $\neg\Delta_i\chi \in s$, and
  - for all ϕ, $\Delta_i\phi \wedge \Delta_i(\chi \to \phi) \in s$ implies $\phi \in t$.
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

See the similarity with the standard canonical definition:

For all $s, t \in S^c$, $sR^c_i t$ iff for all ϕ: $\Box\phi \in s$ implies $\phi \in t$.

Truth lemma

Lemma

For all $\phi \in \mathbf{NCL}$*:* $\mathcal{M}^c, s \vDash \phi \iff \phi \in s$.

Proof idea (for the case of $\Delta\psi$): $\Longrightarrow$: suppose $\Delta_i\psi \notin s$ then $\neg\Delta_i\psi \in s$. To show $\mathcal{M}^c, s \nvDash \Delta_i\psi$ we need to construct **two** i-successors of s that disagree about ψ. Having Lindenbaum lemma in mind, we need to show:

1. $\{\phi \mid \Delta_i\phi \land \Delta_i(\psi \to \phi) \in s\} \cup \{\psi\}$ is consistent.
2. $\{\phi \mid \Delta_i\phi \land \Delta_i(\neg\psi \to \phi) \in s\} \cup \{\neg\psi\}$ is consistent.

1 relies on the validity of the following long formula (proved by using `KwCon` and `KwDis`.) and 2 needs `KwNeg` in addition.

$$\Delta_i(\bigwedge_{j=1}^{k} \phi_j \to \neg\psi) \land \bigwedge_{j=1}^{k} \Delta_i\phi_j \land \bigwedge_{j=1}^{k} \Delta_i(\psi \to \phi_j) \to \Delta_i\psi$$

NCL over other frame classes

Notation	Axiom Schemas	Systems	Frames
KwT	$\Delta_i\phi \wedge \Delta_i(\phi \to \psi) \wedge \phi \to \Delta_i\psi$	$\mathbb{SNCLT} = \mathbb{SNCL} + \texttt{KwT}$	*reflexive*
Kw4	$\Delta_i\phi \to \Delta_i(\Delta_i\phi \vee \psi)$	$\mathbb{SNCL}4 = \mathbb{SNCL} + \texttt{Kw4}$	*transitive*
Kw5	$\neg\Delta_i\phi \to \Delta_i(\neg\Delta_i\phi \vee \psi)$	$\mathbb{SNCL}5 = \mathbb{SNCL} + \texttt{Kw5}$	*euclidean*
wKw4	$\Delta_i\phi \to \Delta_i\Delta_i\phi$	$\mathbb{SNCLS}4 = \mathbb{SNCL} + \texttt{KwT} + \texttt{wKw4}$	*ref.&trans.*
wKw5	$\neg\Delta_i\phi \to \Delta_i\neg\Delta_i\phi$	$\mathbb{SNCLS}5 = \mathbb{SNCL} + \texttt{KwT} + \texttt{wKw5}$	*equivalence*
KwB	$\phi \to \Delta_i((\Delta_i\phi \wedge \Delta_i(\phi \to \psi)$ $\wedge\neg\Delta_i\psi) \to \chi)$	$\mathbb{SNCLB} = \mathbb{SNCL} + \texttt{KwB}$	*symmetric*

Note: $\mathbb{SNCL} + \texttt{wKw4}$ and $\mathbb{SNCL} + \texttt{wKw5}$ are **not** complete over the classes of transitive and euclidean frames respectively.

- We find axioms inspired by AD.
- We manipulate the canonical model.

Example: over reflexive frames

How do we get the axioms? The navie translation of axiom `T` does not work: $\Delta_i\phi \wedge \phi \to \phi$ is simple a tautology. Instead, we use the AD schema to translate $\Box_i$. We start with (a version) of the `T` axiom $\Box_i\neg\phi \to \neg\phi$ and add a precondition $\neg\Delta_i\psi$:

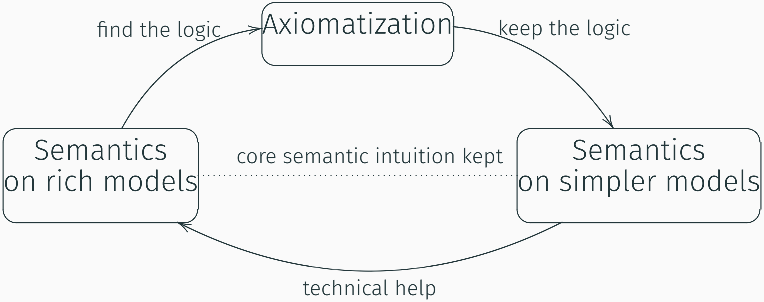
$$
\begin{aligned}
& \neg\Delta_i\psi \to (\Box_i\neg\phi \to \neg\phi) && (1)\\
\Leftrightarrow\ & \neg\Delta_i\neg\psi \wedge \Box_i\neg\phi \to \neg\phi && (2)\\
\Leftrightarrow\ & \neg\Delta_i\neg\psi \wedge \Delta_i\neg\phi \wedge \Delta_i(\neg\psi \to \neg\phi) \to \neg\phi && (3)\\
\Leftrightarrow\ & \Delta_i\phi \wedge \Delta_i(\phi \to \psi) \wedge \phi \to \Delta_i\psi && (4)
\end{aligned}
$$

`KwT` is valid on all the reflexive frames. The Canonical model is not reflexive but we can add the reflexive arrows safely (preserving true lemma). The symmetry case is much more complicated.

Simplify the semantics while keeping the logic

Common difficulties: weak language vs. rich semantics

To restore the balance between the language and model:



Recall the axiomatization

TAUT	all instances of tautologies
KwCon	$\Delta_i(\phi) \wedge \Delta_i(\psi) \to \Delta_i(\phi \wedge \psi)$
KwDis	$\Delta_i\phi \to \Delta_i(\phi \to \psi) \vee \Delta_i(\neg\phi \to \chi)$
KwNeg	$\Delta_i\phi \leftrightarrow \Delta_i\neg\phi$
KwTop	$\Delta_i\top$
MP	From ϕ and $\phi \to \psi$ infer ψ
REKw	From $\phi \leftrightarrow \psi$ infer $\Delta_i\phi \leftrightarrow \Delta_i\psi$

Alternative neighbourhood semantics [Fan SL 17]

A neighborhood model is $\mathcal{M} = \langle W, N, V \rangle$ where $N : W \to 2^{2^W}$ assigns each world a set of subsets of W.

The semantics is given by:

$$\mathcal{M}, w \Vdash \Box\phi \iff [\![\phi]\!]^{\mathcal{M}} \in N(w)$$

We say the neighbourhood model is a **NCL**-model if for all $w \in W$

- $W \in N(w)$
- $N(w)$ is closed under complementation
- $N(w)$ is closed under intersection
- $N(w)$ is closed under supersets or co-supersets: $X, Y, Z \subseteq W, X \in N(w)$ implies $X \cup Y \in N(w)$ or $(W \setminus X) \cup Z \in N(w)$.

We have exactly the same valid formulas over **NCL**-models.

Dynamics

We can add public announcements and event updates into the language **PALNC**:

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Delta_i\phi \mid [\phi]\phi$$

With the usual reduction axiom and the following one we can easily axiomatize **PALNC** over various classes of frames:

$$[\phi]\Delta_i\psi \leftrightarrow (\phi \rightarrow (\Delta_i[\phi]\psi \vee \Delta_i[\phi]\neg\psi))$$

Similar story holds if we introduce the event modality.

Other ways of packing in the literature

- false belief: $\neg\phi \wedge \Box\phi$
- essence: $\phi \to \Box\phi$, accident: $\phi \wedge \neg\Box\phi$ (Pan & Yang, Fan)
- strong non-contingency: $(\phi \to \Box\phi) \wedge (\neg\phi \to \Box\neg\phi)$ (Fan)
- strong belief disagreement: $\Box_1\phi \wedge \Box_2\neg\phi$ (Chen & Pan SL18)
- secret for i: $K_i\phi \wedge K_i \bigwedge_{j\neq i} \neg K_j\phi$ (Xiong, Ågotnes, Zhang 20)
- true belief: $(\phi \wedge B_i\phi)$ (Yang 22), mere belief: $(B_i\phi \wedge \neg K_i\phi) \vee (B_i\neg\phi \wedge \neg K_i\neg\phi)$ (Herzig et. al. 21)
- knowing whether ϕ or ψ: $\Box\phi \vee \Box\psi$ (Aloni, Égré, Jager 09)
- propositional dependency/subservience: $Kw\psi$ implies $Kw\phi$ (Goranko & Kuusisto RSL18, Fan)
- in the neighbourhood setting (Fan, van Ditmarsch ICLA15)
- and many more... Check Jie Fan's recent publications and the incoming talk about ignorance on Sunday, about second order ignorance and s.

What is the general theory of such propositional bundles?

Group notions

- distributed knowing whether (Dw) [Fan, Su, LORI17]
- commonly knowing whether (Cw) [Su thesis 2018, Fan et al 21] studies the 5 possible definitions proposed by Wang.
  1. $C\phi \vee C\neg\phi$
  2. $CEw\phi$ where $Ew\phi$ says everyone knows whether ϕ
  3. $\bigwedge_{k\in\omega} Ew^k\phi$
  4. $\bigwedge_{i\in G} CKw_i\phi \vee C\neg Kw_i\phi$
  5. $\bigwedge_{s\in G^*} Kw_s\phi$ where Kw_s is the abbreviation of $Kw_i ... Kw_j$

Implication: $1 \to 2 \to 3, 4, 5$, but $3, 4, 5$ are independent, over arbitrary models.

Cw^5 is not expressible by $C + K$. It is a highly "strange" operator. How to give a direct semantics? How to axiomatize it?

A related question: ultimate ignorance $\bigwedge_{s\in G^*} \nabla_s\phi$. It is also the ultimate independence in the setting of provability logic.

Alternative axiomatization of common knowledge

An alternative axiomatization of epistemic logic with common knowledge (Herzig & Perrotin AiML2020):

- S5 for K
- S4 for C
- FP_0 $C\phi \to E\phi$
- GFP_0 $CE_w\phi \to C_w\phi$

where $E_w\phi := E\phi \vee E\neg\phi$, $C_w\phi := C\phi \vee C\neg\phi$

“Know about” atoms [Copper et al. 2021]

Language:

$$\phi ::= \alpha \mid \neg\phi \mid \phi \wedge \phi$$

$$\alpha ::= p \mid Kw\alpha \mid Cw\alpha$$

Axiomatization:

- $Kw_i Kw_i \alpha$
- $CwCw\alpha$
- $CwKw_i Kw_i \alpha$
- $Cw\alpha \rightarrow Kw_i \alpha$
- $Cw\alpha \rightarrow CwKw_i \alpha$
- $\bigwedge_{i \in I}(Kw_i \alpha \wedge CwKw_i \alpha) \rightarrow Cw\alpha$

The multi-agent logic is NP-complete (as single-agent S5)! It is enough for many applications.

Knowing whether and non-contingency

There is a large body of research on non-contingency since 1960s involving authors such as Montgomery, Routley, Humberstone, Segerberg, Creswell, Kuhn, Steinsvold and so on.

There are also works on ignorance logic (knowing whether) in epistemic logic.

Surprisingly, the two communities were ignorant about each other's work on such logics! See our RSL article.