Advanced Modal Logic XXIV

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1. Finitary Method: Completeness of PDL

Propositional Dynamic Logic

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid [\pi]\phi$$
$$\pi ::= a \mid \phi? \mid (\pi;\pi) \mid (\pi + \pi) \mid \pi^*$$

where $p \in \mathbf{P}, a \in \mathbf{\Sigma}$. The semantics is defined on Kripke models $\langle W, \{R_a \mid a \in \mathbf{\Sigma}\}, V\rangle$ with relations of *atomic actions* only. The semantics is defined as:

$$\boxed{\mathcal{M}, s \vDash [\pi]\phi \quad \Leftrightarrow \quad \text{for all } t \text{ such that } sR_\pi t : \mathcal{M}, t \vDash \phi}$$

where R_π is computed from those R_a where a is atomic:
$R_{\pi_1;\pi_2} = \{(s,t) \mid \exists u : sR_{\pi_1} u \text{ and } uR_{\pi_2} t\}$; $R_{\pi_1+\pi_2} = R_{\pi_1} \cup R_{\pi_2}$;
$R_{\pi^*} = \bigcup_{n>0} R_{\underbrace{\pi;\ldots;\pi}_{n}} \cup \{(s,s) \mid s \in W\}$, i.e., R_{π^*} is the *reflexive transitive closure* of R_π. $sR_{\psi?}t \iff s = t$ and $\mathcal{M}, s \vDash \psi$.

A proof system

System $\mathbb{S}_{PDL}$

Axioms		Rules	
TAUT	all the instances of tautologies	MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
DISTK	$[\pi](p \to q) \to ([\pi]p \to [\pi]q)$	GEN	$\dfrac{\phi}{[\pi]\phi}$
COMP	$[\pi_1;\pi_2]p \leftrightarrow [\pi_1][\pi_2]p$	SUB	$\dfrac{\phi}{\phi[p/\psi]}$
CHO	$[\pi_1 + \pi_2]p \leftrightarrow [\pi_1]p \wedge [\pi_2]p$		
TEST	$[q?]p \leftrightarrow (q \to p)$		
FIX	$[\pi^*]p \leftrightarrow (p \wedge [\pi][\pi^*]p)$		
IND	$(p \wedge [\pi^*](p \to [\pi]p)) \to [\pi^*]p$		

SUB works for all the formulas (including tests).

PDL is not compact

It is clear that the following set is finitely satisfiable but it is not satisfiable as a whole:

$$\{\langle a^*\rangle p, \neg p, \neg\langle a\rangle p, \neg\langle a; a\rangle p, \neg\langle a; a; a\rangle p, \ldots\}$$

Therefore we cannot expect strong completeness for sure. However, we want to prove the weak completeness which amounts to find a model for each consistent formula.

To simplify the story, we consider the test-free PDL (tfPDL) and system $\mathbb{S}_{tfPDL}$ ($\mathbb{S}_{PDL}$ without the $\mathtt{TEST}$ axiom). Note that, test-free PDL is strictly less expressive compared to the full PDL (why?).

Proof strategy

If we build a standard canonical model for $\mathbb{S}_{tfPDL}$ w.r.t. all the R^c_π defined by:

$$sR^c_\pi t \iff \text{for all } [\pi]\phi \in s, \phi \in t (\text{ equiv. for all } \phi \in t, \langle\pi\rangle\phi \in s)$$

then the following "truth lemma" holds trivially:

$$\mathcal{M}^c, s \Vdash \phi \iff \phi \in s$$

where $\mathcal{M}^c, s \Vdash [\pi]\phi \iff$ for all t such that $sR^c_\pi t$: $\mathcal{M}^c, t \Vdash \phi$. Note that R^c_π is not computed by using R^c_a where a is atomic. However, $\vDash$ does not coincide with $\Vdash$ since there may be R_{π^*} which is not the reflexive transitive closure of R_π (think about how $\{\langle a^*\rangle p, \neg p, \neg\langle a\rangle p, \neg\langle a; a\rangle p, \dots\}$ is "satisfiable" in $\mathcal{M}^c$).

Proof strategy

It is impossible to build a canonical model for the infinite set $\{\langle a^*\rangle p, \neg p, \neg\langle a\rangle p, \neg\langle a;a\rangle p, \ldots\}$ but we can build a canonical model for each finite consistent set of formulas, in particular for a consistent formula ϕ.

Here we recall the filtration technique: we can compress a big model into a smaller one preserving the truth for a subformula-closed finite set of formulas.

Proof strategy

The strategy consists of the following steps:

1. Given a consistent formula, find a suitable closure of it;
2. According to the closure, build a filtration $\mathcal{N}$ from $\mathcal{M}^c$;
3. Prove that $\mathcal{N}, [s] \vDash \psi \iff \mathcal{M}^c, s \Vdash \psi$ for all the ψ in the closure.

Note that, we build the filtration w.r.t. R_a^c in $\mathcal{M}^c$, we discards the information about R_π^c for non-atomic π. Therefore we cannot use the filtration theorem directly, and need to show the filtration really preserves the truth value of the relevant formulas in (3).

Step one: closure

The Fischer-Ladner closure (FL) is defined as follows:

$$
\begin{aligned}
FL(p) &= \{p\} \\
FL(\neg\phi) &= \{\neg\phi\} \cup FL(\phi) \\
FL(\phi \wedge \psi) &= \{\phi \wedge \psi\} \cup FL(\phi) \cup FL(\psi) \\
FL([\pi]\psi) &= FL^{\Box}([\pi]\psi) \cup FL(\psi) \\
FL^{\Box}([a]\psi) &= \{[a]\psi\} \\
FL^{\Box}([\pi_1;\pi_2]\psi) &= \{[\pi_1;\pi_2]\psi\} \cup FL^{\Box}([\pi_1][\pi_2]\psi) \cup FL^{\Box}([\pi_2]\psi) \\
FL^{\Box}([\pi_1+\pi_2]\psi) &= \{[\pi_1+\pi_2]\psi\} \cup FL^{\Box}([\pi_1]\psi) \cup FL^{\Box}([\pi_2]\psi) \\
FL^{\Box}([\pi^*]\psi) &= \{[\pi^*]\psi\} \cup FL^{\Box}([\pi][\pi^*]\psi)
\end{aligned}
$$

Note that $FL^{\Box}$ will not unravel $[\pi^*]\psi$ forever: it stops after unravelling the first $[\pi]$ in $[\pi][\pi^*]\psi$.
The closure is due to the equivalent form of the $[\pi]\phi$: some subformulas are not syntactic subformula, e.g., since $\vdash [\pi^*]p \leftrightarrow (p \wedge [\pi][\pi^*]p)$, $[\pi][\pi^*]$ is a “hidden” subformula of $[\pi^*]\phi$.

Step one: closure

It is not hard to show the following:

Proposition

1. *If* $[\pi]\psi \in FL(\phi)$, *then* $\psi \in FL(\phi)$
2. *If* $[\pi_1;\pi_2]\psi \in FL(\phi)$, *then* $[\pi_1][\pi_2]\psi \in FL(\phi)$
3. *If* $[\pi_1 + \pi_2]\psi \in FL(\phi)$, *then* $[\pi_1]\psi \in FL(\phi)$, $[\pi_2]\psi \in FL(\phi)$
4. *If* $[\pi^*]\psi \in FL(\phi)$, *then* $[\pi][\pi^*]\psi \in FL(\phi)$
5. $Sub(\phi) \subseteq FL(\phi)$

Moreover, $FL(\phi)$ is indeed a finite set:

Proposition

$|FL(\phi)| \leq |\phi|$ *and* $|FL^{\Box}([\pi]\psi)| \leq |\pi|$.

Step two: filtration

Given a formula ϕ, we can define the equivalence relation over $\mathcal{M}^c$: $s \sim t$ iff for all $\psi \in FL(\phi)$: $\psi \in s \iff \psi \in t$.

Given ϕ, the filtration $\mathcal{M}^c_{FL(\phi)}$ is a tuple $\langle W, \{R_a \mid a \in \mathbf{\Sigma}\}, V\rangle$ where:

- $W = \{[s]_\sim \mid s \in \mathcal{M}^c\}$
- $uR_a v$ iff there exists $s \in u$ and $t \in v$ such that $sR^c_a t$ in $\mathcal{M}^c$.
- $p \in V(u)$ iff $p \in V(s)$ for all $s \in u$.

Think about the definition of V.

Step two: filtration

To show:

$$\mathcal{M}^c_{FL(\phi)}, [s] \vDash \phi \iff \mathcal{M}^c, s \Vdash \phi$$

The most crucial step is to prove:

$$\mathcal{M}^c_{FL(\phi)}, [s] \vDash [\pi]\psi \iff \mathcal{M}^c, s \Vdash [\pi]\psi$$

for non-atomic π. For that we can show the two properties of the filtration (for any π):

- If $sR^c_\pi t$ then $[s]R_\pi[t]$;
- If $[s]R_\pi[t]$ and $s \Vdash [\pi]\psi \in FL(\phi)$ then $t \Vdash \psi$.

Step three: $sR^c_\pi t$ then $[s]R_\pi[t]$

Induction on the structure of π: $\pi = a$ is trivial by definition.
$\pi = \pi_1;\pi_2$: if we can find a state u in $\mathcal{M}^c$ such that: $sR^c_{\pi_1}uR^c_{\pi_2}t$, then by induction, we have $[s]R_{\pi_1}[u]R_{\pi_2}[t]$ thus $[s]R_{\pi_1;\pi_2}[t]$. This amounts to showing that the following set is consistent:
$\{\chi \mid [\pi_1]\chi \in s\} \cup \{\langle\pi_2\rangle\psi \mid \psi \in t\}$. Suppose not, then there are a few $[\pi_1]\chi_i \in s$ and some $\psi_j \in t$ such that $\vdash \bigwedge \chi_i \to \neg \bigwedge\langle\pi_2\rangle\psi_j$.
By DISK and NEC we have

$$\vdash [\pi_1]\bigwedge \chi_i \to [\pi_1]\neg \bigwedge\langle\pi_2\rangle\psi_j.$$

Thus $[\pi_1]\neg\bigwedge\langle\pi_2\rangle\psi_j \in s$. Since $\bigwedge \psi_j \in t$ we have $\langle\pi_1;\pi_2\rangle\bigwedge\psi_j \in s$ (by definition of $R^c_{\pi_1;\pi_2}$) which contradicts to $[\pi_1]\neg\bigwedge\langle\pi_2\rangle\psi_j \in s$.
The case of $\pi_1 + \pi_2$ is relatively easy.

Step three: $sR^c_\pi t$ then $[s]R_\pi[t]$

For π^*, the strategy for $\pi_1;\pi_2$ does not work: we may not be able find a R_π path to a $R^c_{\pi^*}$-reachable state (think about a^*).

We make use of the finiteness of the filtration as follows:

1. use a formula to characterize R_{π^*}-reachable states from $[s]$
2. show that if $sR^c_{\pi^*}t$ then t satisfies that formula, thus $[t]$ is reachable from $[s]$.

Step three: $sR^c_\pi t$ then $[s]R_\pi[t]$

Now the formula is

$$\theta_{[s]} = \bigvee_{[s]R_{\pi^*}[t]} \chi_{[t]} \text{ where } \chi_{[t]} = \bigwedge_{\psi\in FL(\phi)\cap t} \psi \wedge \bigwedge_{\psi\in FL(\phi) \text{ but } \psi\notin t} \neg\psi$$

It is not hard to see that $\mathcal{M}^c, u \Vdash \theta_{[s]} \iff [s]R_{\pi^*}[u]$ ($\star$).

Suppose $sR^c_{\pi^*}t$ we need to show $t \Vdash \theta_{[s]}$

for all t_1, t_2, $t_1 \Vdash \theta_{[s]}$ and $[t_1]R_\pi[t_2]$ implies $[s]R_{\pi^*}[t_2]$ (by $\star, R_{\pi^*}$)

$\Rightarrow$ for all t_1, t_2, $t_1 \Vdash \theta_{[s]}$ and $[t_1]R_\pi[t_2]$ implies $t_2 \Vdash \theta_{[s]}$(by $\star$)

$\Rightarrow$ $\mathcal{M}^c \Vdash \theta_{[s]} \to [\pi]\theta_{[s]}$ (by IH for R^c_π and the semantics)

$\Rightarrow$ $\mathcal{M}^c, s \Vdash \theta_{[s]} \wedge [\pi^*](\theta_{[s]} \to [\pi]\theta_{[s]})$

(by the semantics and the fact that $[s]R_{\pi^*}[s]$ and ($\star$))

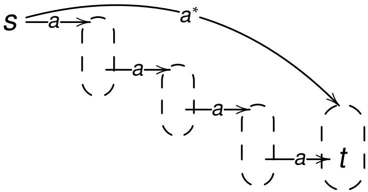
$\Rightarrow$ $\theta_{[s]} \wedge [\pi^*](\theta_{[s]} \to [\pi]\theta_{[s]}) \in s$ (by truth lemma for $\mathcal{M}^c$ w.r.t. $\Vdash$)

$\Rightarrow$ $[\pi^*]\theta_{[s]} \in s$ (by `IND`)

$t \Vdash \theta_{[s]}$ (by definition of $R^c_{\pi^*}$) and the truth lemma for $\Vdash$

Step three: $sR^c_\pi t$ then $[s]R_\pi[t]$

The a^*-path in the filtration model can be restored in $\mathcal{M}^c$ as follows, where the circles denotes the equivalence classes w.r.t. $FL(\phi)$:



Step three: $[s]R_\pi[t]$ & $s \Vdash [\pi]\psi \in FL(\phi) \implies t \Vdash \psi$

Again, proof by induction on the structure of π in $[\pi]\psi$:
$\pi = a$: if $[s]R_a[t]$ then there exist $s' \in [s], t' \in [t]$ such that $s' R^c_a t'$. Now suppose $s \Vdash [a]\psi \in FL(\phi)$ then $[a]\psi \in s$ thus $[a]\psi \in s'$ therefore $\psi \in t'$ thus $\psi \in t$, i.e., $t \Vdash \psi$.
$\pi = \pi_1 ; \pi_2$: if $[s]R_\pi[t]$ then there is a $[v]$ such that $[s]R_{\pi_1}[v]$ and $[v]R_{\pi_2}[t]$. Now suppose $s \Vdash [\pi_1 ; \pi_2]\psi \in FL(\phi)$ we have $s \Vdash [\pi_1][\pi_2]\psi$ by truth lemma and the fact that s is an MCS. Since $[\pi_1][\pi_2]\psi \in FL(\phi)$ and $[s]R_{\pi_1}[v]$, by IH, $v \Vdash [\pi_2]\psi \in FL(\phi)$. By IH again $t \Vdash \psi$.
We omit the case of $\pi_1 + \pi_2$.

Step three: $[s]R_\pi[t]$ & $s \Vdash [\pi]\psi \in FL(\phi) \implies t \Vdash \psi$

π^* : Suppose $[s]R_{\pi*}[t]$ then there is a R_π-path from $[s]$ to $[t]$ via $[s_1]\ldots[s_k]$. Now suppose $s \Vdash [\pi^*]\psi \in FL(\phi)$ then by `FIX` and the truth lemma for $\mathcal{M}^c$ w.r.t. $\Vdash$, $s \Vdash [\pi][\pi^*]\psi \in FL(\phi)$, thus by IH (note that we are doing induction on the outermost $[\pi]$) $s_1 \Vdash [\pi^*]\psi$. Again, $s_1 \Vdash [\pi][\pi^*]\psi$ by `FIX` and the truth lemma. By IH again $s_2 \Vdash [\pi^*]\psi$. We can continue this process and obtain $t \Vdash [\pi^*]\psi$, and thus $t \Vdash \psi$ by axiom `FIX` and the truth lemma again.

Check the textbook for a direct construction of the finite canonical model.