Advanced Modal Logic XXIII

Yanjing Wang

Department of Philosophy, Peking University

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Advanced Modal Logic (2024 Spring)

Step by step method

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What if the canonical model is not in the right shape? In practice, you may leave out “what if” for non-trivial results…

- Massage the model into the right shape.
- Use the canonical model to guide your new construction of the right model.

The detailed transformation techniques may vary from case to case.

In many cases, the properties of the frames cannot be captured by the modal formulas but it might not affect the logic obtained.

Tense logic

The language of the basic *Tense Logic*:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid G\varphi \mid H\varphi$$

$G\varphi$ reads φ always holds in the future (G can be viewed as a box operator). $H\varphi$ expresses that φ always held in the past.

The semantics is defined on Kripke models with a single relation (W, R, V) (alternatively we can include *two* relations which are converse of each other):

$$\mathcal{M}, w \models G\varphi \iff \text{for all } v \text{ such that } wRv \; \mathcal{M}, v \models \varphi$$

$$\mathcal{M}, w \models H\varphi \iff \text{for all } v \text{ such that } vRw \; \mathcal{M}, v \models \varphi$$

We define the dual of G as F (future) and the dual of H as P (past).

Tense logic K_tQ

The basic tense logic contains the following axioms:

K_G	$G(p \to q) \wedge Gp \to Gq$
K_H	$H(p \to q) \wedge Hp \to Hq$
SYM	$p \to GPp$, $p \to HFp$
Rules	NEC_G, NEC_H, MP, $USUB$

The system K_tQ also includes other intuitive axioms:

4	$FFp \to Fp$ (what about $PPp \to Pp$?)
D_r	$Gp \to Fp$
D_l	$Hp \to Pp$
Den	$Fp \to FFp$
$.3r$	$(Fp \wedge Fq) \to F(p \wedge Fq) \vee F(q \wedge Fp) \vee F(p \wedge q)$
$.3l$	$(Pp \wedge Pq) \to P(p \wedge Pq) \vee P(q \wedge Pp) \vee P(p \wedge q)$

Completeness of K_tQ over $\mathbb{Q}$

A DUWTO frame is a frame satisfying the following properties:

- Density
- Unbounded to both directions
- Weakly Total Order:
  - Transitive
  - Trichotomous (any two distinct points are comparable)

Theorem

$\mathsf{K_tQ}$ *is sound and strongly complete over DUWTO frames.*

Proof.

Den, 4, D_r, and D_l are canonical for the corresponding properties. $.3_r$ and $.3_l$ are canonical for the property of no-branching of R and its converse respectively. We just need to make sure trichotomy by considering the generated submodel w.r.t. R and its converse. □

Theorem

$K_t Q$ *is sound and strongly complete over* $(\mathbb{Q}, <)$.

It is not a straightforward corollary of the previous completeness result: the canonical model is not irreflexive and it may have uncountably many worlds.

You can use the bulldozing technique to have a strict total order then use Löwenheim-Skolem theorem to prove that there is a countable model (since the properties we care are first-order definable). However, here we will build the model constructively. *step by step*.

Theorem (Cantor)

Any two countably infinite strictly totally ordered sets that are dense and unbounded (to both directions) are isomorphic.

Proof.

(Sketch of a back-and-forth and step-by step method) Given two countably infinite strictly totally ordered sets that are dense and unbounded $\mathcal{A} = (A, <)$ and $\mathcal{B} = (B, <')$. We enumerate elements in A as $a_0, a_1, \dots$ and enumerate B as $b_0, b_1, \dots$. We gradually build an isomorphisim I between $\mathcal{A}$ and $\mathcal{B}$ step by step by building partial isomorphisms I_k starting from $I_0(a_0) = b_0$, such that a_k is in the domain **and** b_k is in the range, using density, unboundedness and the fact that each partial isomorphism is about finitely many objects. Finally we make the union $I = \bigcup_{k<\omega} I_k$, it is easy to see it is an isomorphism. □

Thus we just need to build a countable dense unbounded strict total order.

The general idea of the proof

We build a countable model by starting from a singleton model and add one state at one step then take the union of the partial models in those countably many steps.

The central idea is to list countably many possible defects (in order to satisfy the targeted maximal consistent set), and resolve them one by one. It is crucial to show that we do not need to *redo* any defect which was resolved in the earlier steps.

When building the partial models, we label the states with maximal consistent sets in the canonical model and use the canonical relation to find the building material to resolve some defects. In the end for the constructed model we have the truth lemma.

Networks

Definition (Network)

A network is a triple $\mathcal{N} = (N, R, \nu)$ such that R is a binary relation on N, and ν assigns each point in N a maximal consistent set.

Definition (Coherent network)

A network $\mathcal{N} = (N, R, \nu)$ is *coherent* if:

- C1 R is a strict total order over N, and
- C2 $\nu(s)R^c\nu(t)$ for all $s, t \in N$ such that sRt, where R^c is the canonical relation.

These are what we want to preserve in every step.

Definition (Saturated network)

A network $\mathcal{N} = (N, R, \nu)$ is *saturated* if:

S1 R is unbounded in both directions, and

S2 R is dense, and

S3 if $F\psi \in \nu(s)$ $(P\psi \in \nu(s))$ for some $s \in N$ then there is some $t \in N$ such that sRt (tRs) and $\psi \in \nu(t)$.

These are what we want to fix at each step. A network is *perfect* if it is both coherent and saturated. A perfect network is clearly a dense unbounded strict ordering.

We can turn a network into a model:

Definition (Induced model)

Given a network $\mathcal{N} = (N, R, \nu)$ the induced model $\mathcal{M}_{\mathcal{N}} = (N, R, V)$ where $p \in V(s) \iff p \in \nu(s)$ for all $s \in N$.

Truth lemma

Lemma

Given a perfect network $\mathcal{N} = (N, R, \nu)$, for all tense logic formulas φ and all points $s \in N$:

$$\mathcal{M}_{\mathcal{N}}, s \models \varphi \iff \varphi \in \nu(s).$$

Proof.

Boolean cases are easy. We only consider the case when $\varphi = G\psi$ ($H\psi$ is similar):

$\Leftarrow$: $\mathcal{M}_{\mathcal{N}}, s \not\models G\psi$ implies that there is a t in $\mathcal{N}$ such that sRt and $\neg\psi \in \nu(t)$ (by IH). Due to C2 we have $\nu(s)R^c\nu(t)$, then by the definition of R^c, $G\psi \notin \nu(s)$.

$\Rightarrow$: By S3, $G\psi \notin \nu(s)$ (namely, $F\neg\psi \in \nu(s)$) implies there is $t \in N$ such that sRt and $\neg\psi \in \nu(t)$. By IH, $\mathcal{M}_{\mathcal{N}}, s \not\models G\psi$. □

Now the remaining task is to build a countably infinite perfect network. Due to Cantor's theorem it is isomorphic to $(\mathbb{Q}, <)$.

Definition (S-defects)

Given a network $\mathcal{N} = (N, R, \nu)$:

- $S1$-defect is a point $s \in N$ such that s does not have any successor or predecessor.
- $S2$-defect is a pair of points $(s, t) \in N \times N$ such that there is no other point u in-between.
- $S3$-defect is a pair of a point s and a formula $F\psi$ (or $P\psi$) such that there is no t such that sRt (tRs) and $\psi \in \nu(t)$.

Lemma

Given a finite coherent network $\mathcal{N} = (N, R, \nu)$ and a defect in it, there is always an extension $\mathcal{N}' = (N', R', \nu')$ of $\mathcal{N}$ (such that $\mathcal{N}$ is a subnetwork of $\mathcal{N}'$) lacking this defect.

Proof.

S1-defect at s: pick a successor Δ of $\nu(s)$ in $\mathcal{M}^c$ (why possible?), and extend $\mathcal{N}$ with a new point t and let $\nu'(t) = \Delta$ and let $R' = R \cup \{(s', t) \mid s'Rs \text{ or } s' = s\}$. The extension is coherent by transitivity of $\mathcal{M}^c$ and the fact that $\mathcal{N}$ is coherent. Similar for the case when lacking the predecessor.
S2-defect for (s, t): pick a Δ in between $\nu(s)$ and $\nu(t)$ in the canonical model, and extend $\mathcal{N}$ with a new point u between s and t with $\nu'(u) = \Delta$. We can show that the extension is still coherent. (To be continued) □

Repair

continues.

S3-defect at s about $F\varphi$ ($P\varphi$ case is similar): We need to insert a φ point at the *right place* after s. Let t be the last point such that sRt or $s = t$ and t does have the S3-defect about $F\varphi$ (there is always such a t since $\mathcal{N}$ is finite). Pick a successor Δ of $\nu(t)$ in $\mathcal{M}^c$ such that $\varphi \in \Delta$. Insert u right after t and let $\nu(u) = \Delta$. We need to show the coherence (C2). The only hard case is about the points beyond u in the extension. Take such a point w, we need to show $\nu(u)R^c\nu(w)$. Since the canonical model is trichotomous for connected points, $\nu(u)R^c\nu(w)$ or $\nu(w)R^c\nu(u)$ or $\nu(u) = \nu(w)$. In the first case we are done. The last case is impossible for otherwise the defect would not be there. Suppose $\nu(w)R^c\nu(u)$, then since $\varphi \in \nu(u)$ we have $F\varphi \in \nu(w)$. However, this is not possible since t is the maximal point beyond which there is no $F\varphi$ defect. □

Step by Step

Given an MCS Γ we construct step by step. We use a set W such that $|W| = \omega$ as our set of possible worlds. We can enumerate the points as $w_0, w_1 \ldots$. We collect all the possible defects by $W \times \{F, P\}$ (S1), $W \times W$ (S2), $W \times (\{F\varphi \mid \varphi \in L\} \cup \{P\varphi \mid \varphi \in L\})$ (S3). Since the language and W are both countable, we can actually enumerate all the possible defects one by one: $D_1, D_2, \ldots$.

We start from a singleton network $\mathcal{N}_0 = (N_0, R_0, \nu_0)$ where $N_0 = \{w_0\}$, R_0 is empty and $\nu_0(w_0) = \Gamma$. For each $\mathcal{N}_k$ we resolve the first defect which has not been resolved (there is always some defect due to the finiteness of $\mathcal{N}_k$) and extend the network to $\mathcal{N}_{k+1}$ by adding w_{k+1}. Note that once resolved the defects will never be introduced again.

Step by Step

Since for each N_k, which is finite, there is always some defect, every w_k will be used eventually. When the relevant points are there in the model, the defects about them will be handled at some stage since we will first handle the existing defects with the minimal index. Since once resolved, the defects will never come back, all the relevant defects will be resolved eventually.

Let $\mathcal{N} = (\bigcup_{k \in \mathbb{N}} N_k, \bigcup_{k \in \mathbb{N}} R_k, \bigcup_{k \in \mathbb{N}} \nu_k)$. $\mathcal{N}$ is a perfect network thus by truth lemma we know that w_0 satisfies Γ, Moreover, it is a countably infinite dense unbounded strict total ordering.