Advanced Modal Logic XXII

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Advanced Modal Logic (2024 Spring)

Transforming the canonical model

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What if the logic is indeed complete but the canonical model is not in the right shape?

- Massage the model into the right shape.
- Use the canonical model to guide your new construction of the right model.

The detailed transformation techniques may vary from case to case. Here we only give two detailed examples using unraveling and bulldozing.

In many cases, the properties of the frames *cannot* be captured by the modal formulas but it might not affect the logic obtained.

Of course, even if the frame properties *are* modally definable, it does not mean that the corresponding proof system is always complete.

S4 over partial orders

Theorem

S4 *is sound and strongly complete over partial orders.*

Partial order: reflexive, transitive and antisymmetric
$\forall x \forall y (xRy \wedge yRx \rightarrow x \approx y)$.

However, antisymmetry is not modally definable (in the basic modal language). The canonical frame is only reflexive and transitive. We want to make it also antisymmetric.

S4 over partial orders

The solution is quite straightforward:

1. unravel the canonical model around a maximal consistent set into a tree.
2. make it reflexive and transitive while preserving truth values of formulas.
3. show that the transformed model is also antisymmetric.

We know that the first step preserves the truth values of the modal formulas (at the root).

The second step relies on the fact that the canonical model is already reflexive and transitive. The third step is easy since there is no xRy and yRx such that $x \not\approx y$ since the model is still in a tree-like shape.

Example: S4 over partial orders

Lemma

Let $\mathcal{M}_\Gamma$ be the unravelling of $\mathcal{M}^c$ around Γ, and let $\mathcal{M}_\Gamma^*$ be the reflexive and transitive closure of $\mathcal{M}_\Gamma$, then $\mathcal{M}_\Gamma^*, \Gamma \equiv_{ML} \mathcal{M}^c, \Gamma$.

Proof.

We define a bounded morphism f between $\mathcal{M}_\Gamma^*$ and the generated submodel of $\mathcal{M}^c$ at Γ such that $f(\langle \Gamma, \ldots, \Delta \rangle) = \Delta$. It is clear that f is surjective. We just need to verify the three conditions of bounded morphism. Note that the canonical model $\mathcal{M}^c$ of $S4$ is reflexive and transitive. We need to use these two properties when proving the forth condition of the bounded morphism due to the reflexive and transitive closure $\mathcal{M}_\Gamma^*$. □

Example: K4.3 over strict total orders

- Strict total order: *irreflexive*, transitive, and *trichotomous* $(\forall x \forall y(xRy \vee yRx \vee x \approx y))$.
- Irreflexivity is not modally definable.
- Axiom .3: $\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p)$ defines the no-branching property which is similar to trichotomy $(\forall x \forall y \forall z(xRy \wedge xRz \to y \approx z \vee yRz \vee zRy))$.
- .3 is also a Sahlqvist formula thus we have the canonicity and completeness of K4.3 over transitive and no-branching frames for free.

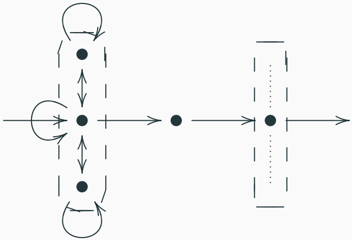
As in the previous case, the indefinable properties do not contribute to the logic:

Theorem

K4.3 is sound and strongly complete over strict total orders.

Example: K4.3 over strict total orders

The canonical model of **K4.3** is in the following shape:



- Fix a maximal consistent set Γ, consider the generated canonical model $\mathcal{M}_\Gamma$ for **K4.3** from Γ.
- "Bulldoze" the reflexive points:
    - Instead of treating the reflexive points individually, we handle *clusters* of the reflexive points (equivalence classes).
    - Totally order the points in each cluster.
    - Introduce countably infinitely many copies of each cluster and order them lexicographically to form a strict total order.

Example: K4.3 over strict total orders

Let the generated submodel of the canonical model from Γ be $\mathcal{M}_\Gamma = (W^c, R^c, V^c)$. We use the index set I to label each cluster (equivalence class). For each C_i we introduce a strict total order $<_i$ among the points in C_i. Then we duplicate each C_i by replacing it with $C_i \times \mathbb{N}$, and define the model $\mathcal{N} = (W, R, V)$ where:

- W is $(W^c \setminus \{C_i \mid i \in I\}) \cup \bigcup_{i \in I}(C_i \times \mathbb{N})$. Let $f : W \to W^c$ be defined such that $f(s)$ is the corresponding world of s in the original canonical model.
- sRt iff one of the following two holds:
  - $f(s)$ and $f(t)$ are in the same cluster (suppose it is C_i) and $s = (w, m)$, $t = (v, n)$:
    - either $m < n$ or $m = n$ and $w <_i v$
  - $f(s)$ and $f(t)$ are not in the same cluster and $f(s)R^c f(t)$

It is not hard to show that f is a surjective bounded morphism from $\mathcal{N}$ to $\mathcal{M}^\Gamma$. In particular Γ is satisfied in $\mathcal{N}$.

The transforming methods vary in different cases

Model transformation is also useful when the semantics is defined differently from the standard one. Examples:

- Universal modality:
  $$\mathcal{M}, w \vDash U\varphi \iff \text{for all } v \in W_{\mathcal{M}}, \mathcal{M}, v \vDash \varphi$$
- Intersection modality:
  $$\mathcal{M}, w \vDash D_{1,2}\varphi \iff \text{for all } v \text{ s.t. } wR_1v \& wR_2v, \mathcal{M}, v \vDash \varphi$$
- Window modality:
  $$\mathcal{M}, w \vDash \boxminus\varphi \iff \text{for all } v \text{ s.t. } \mathcal{M}, v \vDash \varphi, wRv$$

When these are combined with extra frame constraints thing becomes quite complicated.

Other more complicated cases, e.g., non-contingency modality:

$$\mathcal{M}, w \vDash \Box^{nc}\varphi \iff \mathcal{M}, w \vDash \Box\varphi \vee \Box\neg\varphi$$