Advanced Modal Logic XXI

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Frame Incompleteness

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Limitative results

There are normal modal logics that are not canonical.

Theorem

KL is not sound and strongly complete with respect to any class of frames and thus it is not canonical.

Proof.

First note that every frame that makes L valid must be transitive and conversely well-founded. Thus **KL** can only be sound w.r.t. a class of transitive and conversely well-founded frames. Now we take the class $\mathbb{K}$ of *all* such frames. It is clear that if **KL** is not strongly complete w.r.t. $\mathbb{K}$ then it is not strongly complete w.r.t. to any subclass of $\mathbb{K}$: if $\Gamma \vDash_{\mathbb{K}} \varphi$ then $\Gamma \vDash_{\mathbb{K}'} \varphi$ for any subclass $\mathbb{K}'$ of $\mathbb{K}$. However, modal logic **KL** is not compact in $\mathbb{K}$: $\{\Diamond p_0\} \cup \{\Box(p_n \to \Diamond p_{n+1}) \mid 1 \leq n \in \mathbb{N}\}$ which is finitely satisfiable but not satisfiable as a whole in in $\mathbb{K}$. □

Limitative results

We have shown that there are normal modal logics that cannot be sound and *strongly* complete w.r.t. *any* class of frames. However,

Theorem

***KL** is sound and weakly complete with respect to the class of all finite transitive trees.*

What about weak completeness?

There are also normal modal logics which cannot be both sound and (weakly) complete w.r.t. any class of frames.

Incompleteness

Let $\texttt{VB}$ be the formula: $\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$

We will show that $\mathbf{K}\texttt{VB}$ is not sound and (weakly) complete w.r.t. *any* class of frames.

As in the previous proof, we only need to show that $\mathbf{K}\texttt{VB}$ is not sound and (weakly) complete w.r.t. $\mathbb{K} = \{\mathcal{F} \mid \mathcal{F} \vDash \texttt{VB}\}$.

However, we cannot use the compactness argument any more since we are dealing with weak completeness. Therefore we need to find a formula φ such that $\mathbb{K} \vDash \varphi$ but $\nvdash_{\mathbf{K}\texttt{VB}} \varphi$.

Incompleteness

Let $\mathtt{MV}$ be the formula: $\Box\Diamond\top \to \Box\bot$. It is not hard to see that $\mathtt{MV}$ defines the class of frames in which each point is either a dead end or it has a dead end successor. Note that this class of frame is also FO-definable.

We will show the following:

1. $\mathtt{MV}$ and $\mathtt{VB}$ define the same frame class. Thus $\mathbb{K} \models \mathtt{MV}$.
2. $\nvdash_{\mathtt{KVB}} \mathtt{MV}$.

Step 1

$\texttt{VB}$: $\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$
$\texttt{MV}$: $\Box\Diamond\top \to \Box\bot$.

We will show that:
Any frame $\mathcal{F}, w$: $\mathcal{F}, w \vDash \texttt{VB}$ implies $\mathcal{F}, w \vDash \texttt{MV}$ (the other way around also holds).

Proof.

Suppose $\mathcal{F}, w \nvDash \texttt{MV}$ then w is not a dead end and its successors are also not dead ends. Suppose u is a successor of w, let $V(p) = W \backslash \{u\}$. (Do we need to consider the case when u is reflexive?) We can then find a counter-model for $\texttt{VB}$. □

Step 2

$\mathtt{VB}$: $\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$
$\mathtt{MV}$: $\Box\Diamond\top \to \Box\bot$.

We need to show that: $\nvdash_{\mathtt{KVB}} \mathtt{MV}$. How to prove it?

The semantic perspective helps:
We may show the soundness via *an alternative semantics* $\Vdash$:

$$\vdash_{\mathtt{KVB}} \varphi \text{ implies } \mathbb{C} \Vdash \varphi$$

Thus if $\mathbb{C} \nVdash \mathtt{MV}$ we can show $\nvdash_{\mathtt{KVB}} \mathtt{MV}$.

Step 2

Here we just need to consider validity on different structures.

Definition (General frames)

A *general frame* $\mathcal{G}$ is a pair $\langle \mathcal{F}, A \rangle$ where $\mathcal{F} = \langle W, R \rangle$ is a frame and $A \subseteq \mathcal{P}(W)$ is a non-empty collection of *admissible sets* satisfying:

1. $X \in A$ then $W \backslash X \in A$
2. $X, Y \in A$ then $X \cup Y \in A$
3. $X \in A$ then $m_R(X) \in A$

A model based on a general frame is $\langle \mathcal{F}, A, V \rangle$ where V is a valuation such that for any $p \in \mathbf{P}$, $V(p) \in A$. Such valuations are called *admissible*. A formula φ is valid on a general frame $\mathcal{G}$ ($\mathcal{G} \Vdash \varphi$) if for all the admissible valuation V: $\mathcal{G}, V \vDash \varphi$.

What did the conditions of the general frames say?

Step 2

Now consider a specific general frame $\mathcal{G}^\circ = \langle W^\circ, R^\circ, A^\circ \rangle$:

- $W^\circ = \mathbb{N} \cup \{\omega, \omega + 1\}$
- $wR^\circ v \iff \begin{cases} w = \omega + 1 \text{ and } v = \omega \\ w \neq \omega + 1 \text{ and } v < w \end{cases}$
- $A^\circ = \{X \mid \omega \notin X \text{ and } X \text{ is finite }\} \cup \{X \mid \omega \in X \text{ and } X \text{ is co-finite }\}$

You need to show that $\mathcal{G}^\circ$ is indeed a general frame. It is clear that $\mathtt{MV}$ is not valid on $\mathcal{G}^\circ$.

We only need to show that $\vdash_{\mathtt{KVB}} \varphi$ implies $\mathcal{G}^\circ \Vdash \varphi$.

Step 2

To prove the soundness, we can show all the axioms and rules of $\mathtt{KVB}$ are valid on $\mathcal{G}^\circ$.

$\mathbb{K}$: $\mathcal{G}^\circ \Vdash \Box(p \to q) \to (\Box p \to \Box q)$

$\mathtt{MP}$

$\mathtt{NEC}$: $\mathcal{G}^\circ \Vdash \varphi$ implies $\mathcal{G}^\circ \Vdash \Box\varphi$

$\mathtt{USUB}$: $\mathcal{G}^\circ \Vdash \varphi(p)$ implies $\mathcal{G}^\circ \Vdash \varphi[p \backslash \psi]$

We just need to show that:

$$\mathtt{VB} : \mathcal{G}^\circ \Vdash \Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$$

Only need to check $\mathcal{G}^\circ, \omega + 1 \Vdash \Box(\Box(\Box p \to p) \to p)$ (use A°).

Completeness via general frames

Theorem

KVB is not sound and complete w.r.t. any frame class.

Note that `VB` does have a first-order correspondent.

Theorem (See Sec 5.5 of the bluebook)

Any normal modal logic Λ is sound and strongly complete with respect to the class of Λ general frames.

Build the usual canonical model F^c, V^c then..

Let $A^c = \{V^c(\varphi) \mid \varphi \in \mathtt{ML}\}$, i.e. the set of those MCS that are definable within the canonical model. We can show that $\mathcal{F}^c, A^c \Vdash \varphi$ for all $\varphi \in \Lambda$. Suppose not, then then there exist V, Δ and $\varphi \in \Lambda$ such that $\mathcal{F}^c, V, \Delta \nvDash \varphi$. Since for each p_i in φ $V(p_i) \in A^c$, we have $V(p_i) = V^c(\psi_i)$ for some ψ_i, then we can show $\mathcal{F}^c, V^c, \Delta \nvDash \varphi[\psi_i/p_i]$, but $\varphi[\psi_i/p_i]$ is in Λ (thus also in Δ) due to *USUB*, and it is in contradiction with truth lemma. □

Sahlqvist completeness theorem

Theorem

Every Sahlqvist formula is canonical for the first-order property that it defines. Hence, given a set of Sahlqvist axioms Σ*, the logic* $\mathsf{K}\Sigma$ *is strongly complete with respect to the class of frames defined by the first-order correspondents of formulas in* Σ*.*

Note that if a modal formula φ has a FO correspondent α then the class of frames that validate φ is the same as the class of frames that satisfy α.

Proof strategy

We need to show that the frame of canonical model of $\mathsf{K\Sigma}$ validate $\mathsf{K\Sigma}$.

Let $\mathcal{F}^c$ be the frame of the canonical model $\mathcal{M}^c$ of $\mathsf{K\Sigma}$, let A^c be $\{\hat{\varphi} \mid \varphi$ is a modal formulas$\}$ where $\hat{\varphi}$ is the set of maximal consistent sets containing φ (by truth lemma $\hat{\varphi} = V^c(\varphi)$), we call $\langle \mathcal{F}^c, A^c \rangle$ the *canonical general frame* of $\mathsf{K\Sigma}$. It is trivial to show that A^c is indeed an admissible set. The proof strategy is as follows:

- $\langle \mathcal{F}^c, A^c \rangle \vDash \mathsf{K\Sigma}$ (we have just proved).
- $\langle \mathcal{F}^c, A^c \rangle \vDash \mathsf{K\Sigma}$ implies $\mathcal{F}^c \vDash \mathsf{K\Sigma}$

Again, *some* valuations may represent *all* valuations.

$\langle \mathcal{F}^c, A^c \rangle \vDash \mathsf{K}\Sigma$ implies $\mathcal{F}^c \vDash \mathsf{K}\Sigma$

We will show the following:

- $\langle \mathcal{F}^c, A^c \rangle$ is a *descriptive* general frame (to be defined later)
- Sahlqvist formulas are *descriptive-persistent*: if $\langle \mathcal{F}, A \rangle$ is a descriptive general frame then for any Sahlqvist formula φ $\langle \mathcal{F}, A \rangle \vDash \varphi$ implies $\mathcal{F} \vDash \varphi$.

$\langle \mathcal{F}^c, A^c \rangle$ is a *descriptive* general frame

A general frame $\langle \mathcal{F}, A \rangle$ is descriptive if it is:

Differentiated for all $s, t \in \mathcal{F}$: $s = t \iff \forall X \in A, s \in X$ iff $t \in X$.

Tight for all $s, t \in \mathcal{F}$:
$sRt \iff \forall X \in A$: $t \in X$ implies $s \in m_R(X)$

Compact for all $B \subseteq A$: if any finite intersection of B is not empty then $\bigcap B \neq \emptyset$

$\langle \mathcal{F}^c, A^c \rangle$ is a descriptive general frame:

Differentiated $\Delta \neq \Gamma \iff \exists \hat{\varphi} \in A, \Delta \in \hat{\varphi}$ but $\Gamma \notin \hat{\varphi}$.

Tight $\Delta R \Gamma \iff \forall \hat{\varphi} \in A$: $\varphi \in \Gamma$ implies $\Diamond \varphi \in \Delta$

Compact if any finite intersection of B is not empty then $\Theta = \{\psi \mid \hat{\psi} \in B\}$ is consistent for otherwise there is a finite set of B which does not intersect. Then Θ can be extended to a MCS Θ', clearly $\Theta' \in \bigcap B$.

Sahlqvist formulas are descriptive-persistent

Theorem

$\langle \mathcal{F}, A\rangle$ is a descriptive general frame then for any Sahlqvist formula φ: $\langle \mathcal{F}, A\rangle \vDash \varphi$ implies $\mathcal{F} \vDash \varphi$.

Proof.

Please refer to theorem 5.90 on page 321 of the blue book. Here we only sketch the idea. Consider simple Sahlqvist implications in the shape of $\chi \to \psi$ where χ and ψ are both positive formulas. Note that if the minimal valuations that make χ true are all admissible then we are done. However, it is not always the case. □

Sahlqvist formulas are descriptive-persistent

continues.

Now suppose $\langle \mathcal{F}, A \rangle \vDash \chi \to \psi$ and $\mathcal{F}, V_m, s \vDash \chi$, we want to show that $\mathcal{F}, V_m, s \vDash \psi$. Since χ is positive, we know that $\mathcal{F}, V, s \vDash \chi$ for any admissible V extending V_m (notation $V_m \lessdot V$). Since $\langle \mathcal{F}, A \rangle \vDash \chi \to \psi$ and the fact that V is admissible, we have $\mathcal{F}, V, s \vDash \psi$. Since V is arbitrary, we have $s \in \bigcap_{V_m \lessdot V} V(\psi)$. By the properties of the descriptive frame and the exact form of the minimal valuation V_m, we can show that $\bigcap_{V_m \lessdot V} V(p) = V_m(p)$ for all the proposition letters. Furthermore, by analyzing the shape of ψ inductively ($\vee$ and $\diamond$ are the hard parts), we can show that $\bigcap_{V_m \lessdot V} V(\psi) = V_m(\psi)$. thus $\mathcal{F}, V_m, s \vDash \psi$. $\square$

Theorem (Thm 5.56)

The normal modal logic over an elementary class of frames is canonical. (Does it imply Sahlqvist completeness theorem?)