Advanced Modal Logic XX

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May 14th, 2024

Advanced Modal Logic (2024 Spring)

Completeness via Canonicity

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Soundness and completeness

Definition (Soundness)

A modal logic Λ is *sound* with respect to a class $\mathbb{C}$ of frames (or other structures) if for all formulas φ: $\vdash_\Lambda \varphi$ implies $\vDash_\mathbb{C} \varphi$, namely $\Lambda \subseteq \Lambda_\mathbb{C}$.

How to prove the soundness for a Hilbert system?

Definition (Completeness)

A modal logic Λ is *strongly complete* with respect to a class $\mathbb{C}$ of frames (or other structures) if for any set of formulas $\Gamma \cup \{\varphi\}$, if $\Gamma \vDash_\mathbb{C} \varphi$ then $\Gamma \vdash_\Lambda \varphi$. A logic Λ is *weakly complete* with respect to a class $\mathbb{C}$ of frames (or other structures) if for any formula φ, if $\vDash_\mathbb{C} \varphi$ then $\vdash_\Lambda \varphi$, namely $\Lambda_\mathbb{C} \subseteq \Lambda$.

Soundness and completeness

What about "strong soundness" ($\Gamma \vdash_\Lambda \varphi$ implies $\Gamma \vDash_\mathbb{C} \varphi$)? Let's assume the standard soundness: $\vdash_\Lambda \varphi$ implies $\vDash_\mathbb{C} \varphi$. Now suppose $\Gamma \vdash_\Lambda \varphi$ thus there exists $\varphi_0 \dots \varphi_n$ such that $\vdash_\Lambda \varphi_0 \wedge \dots \wedge \varphi_n \to \varphi$. By soundness, $\vDash_\mathbb{C} \varphi_0 \wedge \dots \wedge \varphi_n \to \varphi$, clearly $\Gamma \vDash_\mathbb{C} \varphi$. Therefore strong soundness is equivalent to soundness (unless there are infinitary derivations).

On the other hand, does strong completeness really differ from the weak one? Note that $\Gamma \vDash_\mathbb{C} \varphi$ cannot be reduced to $\Delta \vDash_\mathbb{C} \varphi$ for some finite Δ in general, e.g., $\{\Diamond^n \top \mid n \in \mathbb{N}\} \vDash_\mathbb{C} \bot$ with $\mathbb{C}$ being the collection of finite trees.

In general if the logic is not compact within $\mathbb{C}$ then we cannot expect the logic to be sound and strong complete w.r.t. $\mathbb{C}$ (why?). On the other hand, weak completeness+compactness leads to strong completeness (with classical boolean connectives).

Soundness and completeness

Proposition

A modal logic Λ is strongly complete with respect to a class of structures $\mathbb{C}$ iff every Λ-consistent set of formulas is satisfiable on some $\mathbb{C}$-structure. Λ is weakly complete with respect to a class of structures $\mathbb{C}$ iff every Λ-consistent formula is satisfiable on some $\mathbb{C}$-structure.

for the case of strong completeness.

$\Rightarrow$: Suppose $\Gamma \vDash_{\mathbb{C}} \varphi \implies \Gamma \vdash_{\Lambda} \varphi$ for any $\Gamma \cup \{\varphi\}$. It is clear that if Δ is Λ-consistent then $\Delta \nvdash_{\Lambda} \bot$. Thus $\Delta \nvDash_{\mathbb{C}} \bot$, namely, Δ is satisfiable in $\mathbb{C}$. $\Leftarrow$: Suppose there is a set $\Gamma \cup \{\varphi\}$ such that $\Gamma \vDash_{\mathbb{C}} \varphi$ but $\Gamma \nvdash_{\Lambda} \varphi$. Now $\Gamma \cup \{\neg\varphi\}$ is Λ-consistent (why?) and $\Gamma \cup \{\neg\varphi\}$ is not satisfiable in $\mathbb{C}$. $\square$

Maximal consistent sets (MCS)

Definition (Λ-MCSs)

Let Λ be a modal logic. A set of formulas Γ is maximal Λ-consistent if it is Λ-consistent, and any set of formulas properly containing Γ is Λ-inconsistent. If Γ is a maximal Λ-consistent set of formulas then we say it is a Λ-MCS.

A Λ-MCS Γ has the following properties:

- $\Lambda \subseteq \Gamma$
- if $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ then $\psi \in \Gamma$
- $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$
- $\varphi \vee \psi \in \Gamma \iff \varphi \in \Gamma$ or $\psi \in \Gamma$
- $\varphi \wedge \psi \in \Gamma \iff \varphi \in \Gamma$ and $\psi \in \Gamma$

Lindenbaum's Lemma

Lemma

Every Λ-consistent set of formulas can be extended into a Λ-MCS.

Proof.

Use Zorn's lemma, like in the case of ultrafilter theorem. (not necessarily for countable languages: we can always add the formulas one by one). $\square$

Canonical model for normal modal logics

Definition

The canonical model $\mathcal{M}^\Lambda$ for a normal modal logic Λ is the triple $(W^\Lambda, R^\Lambda, V^\Lambda)$ where:

- W^Λ is the set of all the Λ-MCSs.
- $wR^\Lambda v$ iff (for all φ: $\Box\varphi \in w$ implies $\varphi \in v$)
- $V^\Lambda(p) = \{w \mid p \in w\}$

(W^Λ, R^Λ) is called the canonical frame for Λ.

Note that $wR^\Lambda v$ iff (for all φ: $\varphi \in v$ implies $\Diamond\varphi \in w$).

Truth Lemma and existence lemma

Lemma

For any normal modal logic Λ *and any formula* φ*,*
$\mathcal{M}^\Lambda, w \vDash \varphi \iff \varphi \in w$.

To prove this we need an existence lemma:

Lemma

For any normal modal logic Λ *and any state* $w \in W^\Lambda$*, if* $\neg\Box\varphi \in w$ *then there is a state* $v \in W^\Lambda$ *such that* $wR^\Lambda v$ *and* $\neg\varphi \in v$*.*

Proof.

We need to construct a v from a Λ-consistent set $\{\neg\varphi\} \cup \{\psi \mid \Box\psi \in w\}$. The crucial thing is to show that set is consistent. □

Canonical Model Theorem

Recall that:

Proposition

A logic Λ is strongly complete with respect to a class of structures $\mathbb{C}$ iff every Λ-consistent set of formulas is satisfiable on some $\mathbb{C}$-structure. Λ is weakly complete with respect to a class of structures iff every Λ-consistent formula is satisfiable on some $\mathbb{C}$-structure.

Theorem (Canonical Model Theorem)

Any normal modal logic is strongly complete with respect to its canonical model.

Where do we use the normality? Every axiom should play some role, otherwise...

Completeness via Canonicity

Theorem

K is sound and strongly complete w.r.t. the class of all frames.

What about others?

T	$p \to \Diamond p$	reflexivity
4	$\Diamond\Diamond p \to \Diamond p$	transitivity
B	$p \to \Box\Diamond p$	symmetry
D	$\Box p \to \Diamond p$	seriality

Based on these axioms we have: K4, T (KT), B (KB), KD, S4 (KT4), S5 (KTB4) ...

Cube of normal modal logics (by Liu Yang)

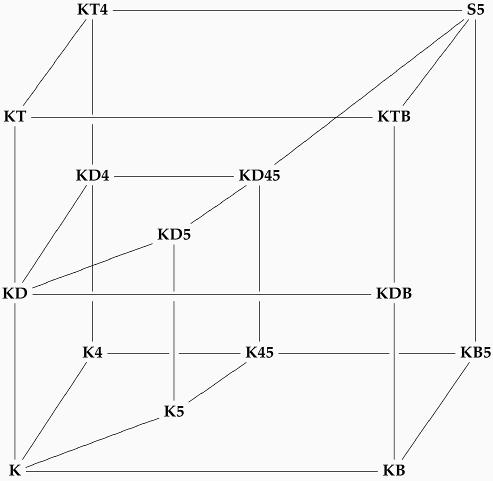

Figure 1. *Relative Strength of Normal Modal Systems Obtained by Combing with* **K** *Various Axioms of* **D**, **T**, **B**, **4**, *and* **5**.

Applications

K4 is sound and strong complete w.r.t. the class of transitive frames.

T is sound and strong complete w.r.t. the class of reflexive frames.

KB is sound and strong complete w.r.t. the class of symmetric frames.

KD is sound and strong complete w.r.t. the class of serial frames. (consider $\Box\top \to \Diamond\top$)

S4 is sound and strongly complete w.r.t. the class of reflexive and transitive frames.

S5 is sound and strongly complete w.r.t. the class of frames with equivalence relations.

Canonicity

Definition (Canonicity)

A formula φ is *canonical* if, for any normal logic Λ , $\varphi \in \Lambda$ implies that φ is valid on the canonical frame for Λ. A normal logic is canonical if its canonical frame is a frame for Λ. If the canonical frame for any normal logic Λ containing φ has the property P and φ is valid on any class of frames with property P then φ is canonical for P.

Theorem (Sahlqvist Completeness Theorem)

Every Sahlqvist formula is canonical for the first-order property it defines. Hence, given a set of Sahlqvist axioms Γ, the logic $K\Gamma$ is sound and strongly complete with respect to the class of frames with the corresponding first-order properties.