Advanced Modal Logic XIX

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Advanced Modal Logic (2023 Spring)

Completeness

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A traditional abstract view of Logic

A logic Λ is a collection of formulas in a formal language.

Syntactic characterization: Λ_S (the collection of formulas that are *provable* in a deductive system S)

Semantic characterization: $\Lambda_{\mathbb{C}}$ (the collection of formulas that are *valid* in a class $\mathbb{C}$ of structures)

A natural question is to ask whether $\Lambda_S = \Lambda_{\mathbb{C}}$ (soundness and (weak) completeness). In practice, we may go from Λ_S to $\Lambda_{\mathbb{C}}$ or the other way around.

Deductive systems

- Hilbert-style system (Euclid, Frege, Hilbert, Russell)
- Natural deduction (Gentzen, Jaškowski, Fitch)
- Sequent calculus (Gentzen)
- Resolution systems, Tableaux...

Modal Logic: a syntactic perspective

Definition (Modal Logics)

A (propositional) modal logic Λ is a set of modal formulas that contains all propositional tautologies and is closed under *modus ponens* and *uniform substitution*. We say that φ is a *theorem* of Λ ($\vdash_\Lambda \varphi$) if $\varphi \in \Lambda$. We say Λ_2 is an *extension* of Λ_1 iff $\Lambda_1 \subseteq \Lambda_2$. Λ is consistent if for any φ: φ and $\neg\varphi$ do not both belong to Λ.

Given a set of formula Γ, is there a minimal modal logic containing Γ? Yes, hint: there is a largest modal logic and the intersection of modal logics is still a modal logic. Therefore we can define the modal logic *generated* from Γ to be the minimal modal logic containing Γ. Uniform substitution is not always necessary for modal logic (even over frames).

Normal modal logic

Definition (Normal modal logic)

A modal logic Λ is a *normal* modal logic if it contains the following formula:

$\texttt{K(ripke)}$: $\quad \Box(p \to q) \to (\Box p \to \Box q)$

and it is closed under:

$\texttt{NEC}$: if $\vdash_\Lambda \varphi$ then $\vdash_\Lambda \Box\varphi$.

There is always a minimal normal modal logic containing Γ (we call it the normal modal logic generated/axiomatized by Γ). The normal modal logic generated by $\emptyset$ is called K. If Γ is non-empty then the normal modal logic generated by Γ is denoted as $\boldsymbol{K\Gamma}$.

What if the primitive modality is $\Diamond$?

Normal modal logic

Definition (normal modal logic of arbitrary similarity types)

A modal logic Λ is a normal modal logic if it contains the formulas:

K^i_∇: $\nabla(r_1, \ldots, r_{i-1}, p \to q, r_{i+1}, \ldots, r_n) \to (\nabla(r_1, \ldots, r_{i-1}, p, r_{i+1}, \ldots, r_n) \to \nabla(r_1, \ldots, r_{i-1}, q, r_{i+1}, \ldots, r_n))$

(where r_i, p, q are distinct proposition letters) and it is closed under generalization:

$\vdash_\Lambda \varphi \implies \vdash_\Lambda \nabla(\ldots, \varphi, \ldots)$ where $\ldots$ stands for arbitrary formulas.

The $\nabla(\bot, \ldots, \varphi, \ldots, \bot)$ version of generalization rule is problematic for completeness (Ex.).

$\boldsymbol{K\Gamma}$ can be represented as $\Lambda_{S_{\boldsymbol{K\Gamma}}}$ where $S_{\boldsymbol{K\Gamma}}$ is a Hilbert-style proof system containing: all the propositional tautologies, the K axiom, and formulas in Γ as axioms, and it has the following inference rules: MP, USUB, NEC corresponding to the closure properties.

MP given $\varphi \to \psi$ and φ, prove ψ

USUB given $\varphi(p)$, prove $\varphi[\psi/p]$ ($\vdash_{S_{\boldsymbol{K\Gamma}}} \varphi(p) \implies \vdash_{S_{\boldsymbol{K\Gamma}}} \varphi[\psi/p]$)

NEC given φ, prove $\Box\varphi$ ($\vdash_{S_{\boldsymbol{K\Gamma}}} \varphi \implies \vdash_{S_{\boldsymbol{K\Gamma}}} \Box\varphi$)

We write $\vdash_{\mathsf{S}} \varphi$ if there is an S-proof with φ being the last item and call φ a *theorem* of S.

It is not hard to show that $\boldsymbol{K\Gamma} = \{\varphi \mid \vdash_{S_{\boldsymbol{K\Gamma}}} \varphi\}$

A simplified example of a proof

$$\vdash_{\mathsf{K}} \Box(p \wedge q) \rightarrow \Box p \wedge \Box q$$

1	$\vdash_{\mathsf{K}} p \wedge q \rightarrow p$	TAUT
2	$\vdash_{\mathsf{K}} \Box(p \wedge q \rightarrow p)$	NEC
3	$\vdash_{\mathsf{K}} \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$	K
4	$\vdash_{\mathsf{K}} \Box(p \wedge q \rightarrow p) \rightarrow \Box(p \wedge q) \rightarrow \Box p$	USUB
5	$\vdash_{\mathsf{K}} \Box(p \wedge q) \rightarrow \Box p$	MP(4, 2)
6	$\vdash_{\mathsf{K}} \Box(p \wedge q) \rightarrow \Box q$	Repeat 1-5 for $p \wedge q \rightarrow q$
7	$\vdash_{\mathsf{K}} \Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$	TAUT

What about

$$\vdash_{\mathsf{K}} (\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$$

Admissible rules

A rule is *admissible* w.r.t. a system if adding this rule to the system does not increase the deductive power of the system (no new theorems).

Some inference rules are derivable by using only the axioms and rules in the system: e.g., $\vdash_{\mathsf{K}} \varphi \to \psi \implies \vdash_{\mathsf{K}} \Box\varphi \to \Box\psi$

1	$\vdash_{\mathsf{K}} \varphi \to \psi$	*Hypothesis*
2	$\vdash_{\mathsf{K}} \Box(\varphi \to \psi)$	`NEC`
3	$\vdash_{\mathsf{K}} \Box(p \to q) \to (\Box p \to \Box q)$	`K`
4	$\vdash_{\mathsf{K}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	`USUB`(3)
5	$\vdash_{\mathsf{K}} \Box\varphi \to \Box\psi$	`MP`(3, 5)

Admissible rules

$\vdash_K \varphi \to \psi \implies \vdash_K \Diamond\varphi \to \Diamond\psi$

1	$\vdash_K \varphi \to \psi$	*Hypothesis*
2	$\vdash_K \neg\psi \to \neg\varphi$	TAUT
3	$\vdash_K \Box\neg\psi \to \Box\neg\varphi$	*above rule*
4	$\vdash_K \neg\Box\neg\varphi \to \neg\Box\neg\psi$	TAUT

There are also admissible rules which cannot be derived by only using axioms and inference rules like the above.

E.g., The empty rule $\vdash_K \Diamond\varphi \implies \vdash_K \varphi$, as another example: $\vdash_K \Box\varphi \implies \vdash_K \varphi$.

Alternative proof system

Instead the K axiom and the rule of necessitation we can have:

$$\begin{array}{rl} \text{C} & \Box p \wedge \Box q \to \Box(p \wedge q) \\ \text{NT} & \Box\top \\ \text{MONO} & \text{given } \varphi \to \psi\text{, prove } \Box\varphi \to \Box\psi \\ & (\vdash_{S_{K\Gamma}} \varphi \to \psi \implies \vdash_{S_{K\Gamma}} \Box\varphi \to \Box\psi) \end{array}$$

We can show that K-axiom is provable and necessitation is admissible (and derivable).

We can weaken C to obtain *weakly aggregative logics*, for example:

$$K_2 : \Box p \wedge \Box q \wedge \Box r \to \Box((p \wedge q) \vee (p \wedge r) \vee (q \wedge r))$$

Which semantics makes this valid but not C?

Important axioms and corresponding extensions

T	$\Box p \to p$	$\mathsf{T} = \mathsf{K}\mathtt{T}$
D	$\Box p \to \Diamond p$	$\mathsf{D} = \mathsf{K}\mathtt{D}$
4	$\Box p \to \Box\Box p$	$\mathsf{S4} = \mathsf{K}\mathtt{T4}$
E	$\Diamond p \to \Box\Diamond p$	$\mathsf{S5} = \mathsf{K}\mathtt{TE}$
B	$p \to \Box\Diamond p$	$\mathsf{B} = \mathsf{K}\mathtt{B}$
L	$\Box(\Box p \to p) \to \Box p$	$\mathsf{GL} = \mathsf{K}\mathtt{L}$

Admissible rules may not be preserved under extensions!

$$\vdash_{\mathsf{K}} \Diamond\varphi \implies \vdash_{\mathsf{K}} \varphi$$

but:

$$\vdash_{\mathsf{T}} \Diamond(p \to \Box p) \not\Rightarrow \vdash_{\mathsf{T}} p \to \Box p$$

Try to prove $\vdash_{\mathsf{K}\mathtt{T}} \Box(p \wedge \neg\Box p) \to \bot$ and $\vdash_{\mathsf{K}\mathtt{D4}} \Box(p \wedge \neg\Box p) \to \bot$ Not all the true beliefs are knowable!

Deduction with assumptions

We only talked about proofs but when do we say a formula is *deducible* from a set of assumptions? (Namely, how to define $\Gamma \vdash_{\mathsf{S}} \varphi$?)

Recall propositional logic: $\Gamma \vdash \varphi$ iff there is a proof of φ based on the assumptions from Γ and the axioms.

Does it work here for modal logic?

Syntactic Consequence via proof system $\vdash_S$

Under such (informal) definition, we do not have the *deduction theorem*: $\{p\} \vdash_K \Box p$ but $\nvdash_K p \to \Box p$. Moreover, $\{p\} \vdash_K q$, but $\nvdash_K p \to q$. What went wrong ? `NEC` and `USUB` may not be applied to φ if $\nvdash_K \varphi$! To fix it, we have the following options:

1. keep these rules but change the definition of $\Gamma \vdash_S \varphi$: $\Gamma \vdash_S \varphi$ iff $\vdash_S \varphi$ or there are $\varphi_0, \dots, \varphi_n \in \Gamma$ such that $\vdash_S (\varphi_0 \wedge \cdots \wedge \varphi_n) \to \varphi$.
2. make `USUB` and `NEC` conditional on $\vdash_S$ and take the usual definition of $\Gamma \vdash_S \varphi$
3. replace all the inference rules by explicit rules for $\Gamma \vdash_S \varphi$

Syntactic Consequence $\vdash_\Lambda$

We may employ the following explicit rules for reasoning about $\Gamma \vdash_S \varphi$:

- $\varphi \in \Gamma \implies \Gamma \vdash_S \varphi$
- φ is an axiom instance $\implies \Gamma \vdash_S \varphi$
- $\Gamma \vdash_S \varphi, \Delta \vdash_S \varphi \to \psi \implies \Gamma \cup \Delta \vdash_S \psi$
- $\emptyset \vdash_S \varphi \implies \Gamma \vdash_S \Box\varphi$

Raul Hakli, Sara Negri. *Does the deduction theorem fail for modal logic?* Synthese (2011).

Given the notion of deduction, we say a set of formulas Γ is *S*-consistent if $\Gamma \nvdash_S \bot$, and Λ-inconsistent otherwise.

If $\Gamma \cup \{\varphi\}$ is a set of formulas then when do we say that φ is a semantic consequence of Γ ($\Gamma \vDash_{\mathbb{C}} \varphi$) where $\mathbb{C}$ is a class of frames?

We have several options: $\Gamma \vDash_{\mathbb{C}} \varphi$ iff

1. for all $\mathcal{F}$ from $\mathbb{C}$: $\mathcal{F} \vDash_{\mathbb{C}} \Gamma \implies \mathcal{F} \vDash_{\mathbb{C}} \varphi$
2. for all $\mathcal{F}, w$ from $\mathbb{C}$: $\mathcal{F}, w \vDash_{\mathbb{C}} \Gamma \implies \mathcal{F}, w \vDash_{\mathbb{C}} \varphi$
3. for all $\mathcal{M}$ based on frames in $\mathbb{C}$: $\mathcal{M} \vDash_{\mathbb{C}} \Gamma \implies \mathcal{M} \vDash_{\mathbb{C}} \varphi$
4. for all $\mathcal{M}, w$ based on frames in $\mathbb{C}$:
   $\mathcal{M}, w \vDash_{\mathbb{C}} \Gamma \implies \mathcal{M}, w \vDash_{\mathbb{C}} \varphi$

For (1) and (2) we have $p \vDash_{\mathbb{C}} q$. (3) and (4) seem OK. Are they the same? Under the definition of (3): $p \vDash_{\mathbb{C}} \Box p$ while for (4) it is not the case. (3) is global (denoted as $\vdash^{g}_{\mathbb{C}}$) and (4) is local (denoted as $\vDash^{l}_{\mathbb{C}}$ or simply $\vDash_{\mathbb{C}}$).

Semantic Consequence

To match $\Gamma \vdash \varphi$, we take the local semantic consequence: for all $\mathcal{M}, w$ based on $\mathbb{C}$, $\mathcal{M}, w \vDash_{\mathbb{C}} \Gamma \implies \mathcal{M}, w \vDash_{\mathbb{C}} \varphi$.

From the semantic point of view, the inference based on assumptions should be truth preserving: if the assumptions are *true* (not valid) then the consequence should be true too. However, `USUB` and `NEC` do not preserve truth (but they do preserve validity) while `MP` preserves truth and validity.

Therefore extra axioms $\neq$ extra assumptions.