Advanced Modal Logic XVIII

Yanjing Wang

Department of Philosophy, Peking University

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Advanced Modal Logic (2024 Spring)

Frame definability

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Frame definability: finite transitive frame

Definition (Jankov-Fine formulas)

Given a finite transitive frame $\mathcal{F}$ we enumerate the states of $\mathcal{F}$ as $w_0, \ldots, w_n$ and associate each state with a distinct proposition letter p_i. The Jankov-Fine formula $\varphi_{\mathcal{F}_{w_i}}$ for $\mathcal{F}, w_i$ is the conjunction of the following formulas:

- p_i
- $\bigoplus_{\{0 \leq i \leq n\}} p_i \wedge \Box \bigoplus_{\{0 \leq i \leq n\}} p_i$ (where $\oplus$ is the exclusive OR)
- $\bigwedge_i (p_i \to \bigwedge_j \{\Diamond p_j \mid w_i \to w_j\}) \wedge \Box \bigwedge_i (p_i \to \bigwedge_j \{\Diamond p_j \mid w_i \to w_j\})$
- $\bigwedge_i (p_i \to \bigwedge_j \{\neg \Diamond p_j \mid w_i \not\to w_j\}) \wedge \Box \bigwedge_i (p_i \to \bigwedge_j \{\neg \Diamond p_j \mid w_i \not\to w_j\})$

$\varphi_{\mathcal{F}_{w_i}}$ "describes" the pointed frame $\mathcal{F}, w_i$.

Frame definability: Finite transitive frame

Proposition

Let $\mathcal{F}$ be a finite transitive point-generated frame (say from w). Then for any transitive frame $\mathcal{F}'$: $\varphi_{\mathcal{F},w}$ is satisfiable at w' in $\mathcal{F}'$ if and only if there exists a surjective bounded morphism from w'-generated subframe of $\mathcal{F}'$ to $\mathcal{F}$.

Frame definability: Finite transitive frame

Proof.

$\Rightarrow$: if $\varphi_{\mathcal{F},w}$ is satisfiable at $\mathcal{F}', V', w'$ for some V' and w', then let W' be the set of states of the w'-generated subframe of F' and let $f : W' \to W$ be defined as $f(v') = v$ such that $V'(v') = \{p_j \mid v = w_j\}$ (this is a singleton set). It is not hard to show that f is a surjective bounded morphism (over frames). Note that you need both the $\Diamond$ part and $\neg\Diamond$ part in $\varphi_{\mathcal{F},w}$.

$\Leftarrow$: If there is a surjective bounded morphism f from w'-generated subframe of $\mathcal{F}'$ to $\mathcal{F}$ then let $V'(v') = \{p_j \mid f(v') = w_j\}$. We can show $\mathcal{F}', V', w'$ satisfies $\varphi_{\mathcal{F},w}$ by preservation of modal truth under bounded morphic images (over models). $\square$

Frame definability: Finite transitive frame

Theorem

Let $\mathbb{K}$ be a class of frames. Then $\mathbb{K}$ is definable by a set of modal formulas within the class of finite transitive frames if and only if it is closed under taking (finite) disjoint unions, generated subframes, and bounded morphic images.

Recall the definition of relative definability: A class of frames $\mathbb{K}$ is definable by Σ w.r.t. class $\mathbb{C}$ if for any $\mathcal{F} \in \mathbb{C}$: $\mathcal{F} \in \mathbb{K} \iff \mathcal{F} \models \Sigma$.

We *do not* need ultrafilter extension here!

Frame definability: Finite transitive frame

$\Leftarrow$ Suppose $\mathbb{K}$ has the desired closure properties.

Let $Th(\mathbb{K}) = \{\varphi \mid \mathbb{K} \vDash \varphi\}$. NTS: For any finite transitive frame $\mathcal{F}$: $\mathcal{F} \vDash Th(\mathbb{K}) \implies \mathcal{F} \in \mathbb{K}$. We need to link $\mathcal{F}$ with *some* frame in $\mathbb{K}$. The ingredients are provided by the previous proposition. Here are two cases: (1) If $\mathcal{F}$ is w-generated then clearly $\varphi_{\mathcal{F},w}$ is satisfiable at $\mathcal{F}, w$ thus $\neg\varphi_{\mathcal{F},w} \notin Th(\mathbb{K})$. Therefore there is a frame $\mathcal{F}'$ in K which satisfies $\varphi_{\mathcal{F},w}$, thus there is a bounded morphism from a generated subframe of $\mathcal{F}'$ to $\mathcal{F}$. By the closure properties $\mathcal{F} \in \mathbb{K}$. (2) If $\mathcal{F}$ is not a pointed generated frame then all its point-generated subframes are in $\mathbb{K}$ (since they validate $Th(\mathbb{K})$). Note that there is a natural surjective bounded morphism from the disjoint union of all such point-generated subframes to $\mathcal{F}$. By the clousure properties $\mathcal{F} \in \mathbb{K}$.

$\square$

Frame definability: Finite transitive frame

Wait! Is $\Rightarrow$ really trivial??

Theorem (problematic in the earlier version of the textbook)

Let $\mathbb{K}$ be a class of frames. Then $\mathbb{K}$ is definable by a set of modal formulas within the class of finite transitive frames if and only if it is closed under taking (finite) disjoint unions, generated subframes, and bounded morphic images.

A class of frames $\mathbb{K}$ is definable by Σ w.r.t. class $\mathbb{C}$ if for any $\mathcal{F} \in \mathbb{C}$: $\mathcal{F} \in \mathbb{K} \iff \mathcal{F} \vDash \Sigma$. If $\mathbb{K}$ does not contain any finite transitive frame, then it is relatively definable by $\bot$!

Theorem (fixed version)

Let $\mathbb{K}$ be a class of finite transitive frames. Then $\mathbb{K}$ is definable by a set of modal formulas within the class of finite transitive frames if and only if it is closed under taking (finite) disjoint unions, generated subframes, and bounded morphic images.

Goldblatt-Thomason Theorem

Which first-order definable frame classes are modally definable (conditional definability)?

Theorem (Goldblatt-Thomason Theorem)

A first-order definable class $\mathbb{K}$ of frames is definable by a set of modal formulas if and only if it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions.

(No requirements for the complement?)Proof strategy (similar to the proof of the previous theorem):

Let $Th(\mathbb{K}) = \{\varphi \mid \mathbb{K} \vDash \varphi\}$. NTS any frame $\mathcal{F} \vDash Th(\mathbb{K}) \implies \mathcal{F} \in \mathbb{K}$.

We need to link $\mathcal{F}$ with *some* frame(s) in $\mathbb{K}$ by the frame construction methods under which $\mathbb{K}$ is closed.

W.l.o.g we assume that $\mathcal{F}$ is generated from w (why?).

Goldblatt-Thomason Theorem

Step 1: Based on $\mathcal{F}$, find some $\mathcal{G}$ in $\mathbb{K}$. Note that we cannot finitely "describe" $\mathcal{F}$ any more.

Let $\Delta = \{\varphi \in \mathtt{ML}^+ \mid \mathcal{F}, V, w \vDash \varphi\}$ where $\mathtt{ML}^+$ extends $\mathtt{ML}$ by (uncountably many) new proposition letters p_A for each $A \subseteq W_{\mathcal{F}}$ and we let $V(p_A) = A$. We can show that Δ is satisfiable in $\mathbb{K}$ based on the fact that Δ is finitely satisfiable (why? we can rewrite each finite set using formulas in $\mathtt{ML}$) in $\mathbb{K}$ and that $\mathbb{K}$ is closed under (frame) ultraproducts (since it is first-order definable).

Therefore there is a frame $\mathcal{G}$ in $\mathbb{K}$ such that $\mathcal{G}, V', v$ satisfies Δ for some V' and v. W.l.o.g we assume that $\mathcal{G}$ is v-generated (why?).

Goldblatt-Thomason Theorem

Step 2: Try to link $\mathcal{F}$ to $\mathcal{G}$. We will show that $\mathfrak{ue}(\mathcal{F})$ is a bounded morphic image of a *countably saturated* ultrapower of $\mathcal{G}$. The key idea is to use the fact that modally equivalent m-saturated models are bisimilar to each other. Then we transfer the bisimulation at the model level to the bounded morphism on the frame level.

We first prove a handy claim:

$\forall \varphi \in \mathtt{ML}^+ : \mathcal{F}, V \vDash \varphi \iff \prod_U (\mathcal{G}, V') \vDash \varphi$
$\mathcal{F}, V \vDash \varphi \iff \mathcal{F}, V, w \vDash \{\Box^n \varphi \mid n \in \mathbb{N}\}$
$\iff \mathcal{G}, V', v \vDash \{\Box^n \varphi \mid n \in \mathbb{N}\} \iff \mathcal{G}, V' \vDash \varphi \iff \prod_U (\mathcal{G}, V') \vDash \varphi$
(think about the Łoś theorem for $\mathtt{ML}$)

Goldblatt-Thomason Theorem

Now we define the morphism $f : W_{\prod_U(\mathcal{G},V')} \to W_{\mathfrak{ue}(\mathcal{F},V)}$ by letting $f(s) = \{A \mid \prod_U(\mathcal{G}, V'), s \vDash p_A\}$. Namely, $f(s) = u$ iff $u = \{A \mid \prod_U(\mathcal{G}, V'), s \vDash p_A\}$.

We NTS: (1) $f(s)$ is indeed an ultrafilter. (2) f is a bounded morphism (3) f is surjective.

(1) is easy (intuitively, an ultrafilter is a maximal consistent set of all the potential Boolean formulas). To prove it formally we do need the previous claim. For example, if $A \subseteq B$ then $p_A \to p_B$ is valid in $\mathcal{F}, V$ thus $\prod_U(\mathcal{G}, V') \vDash p_A \to p_B$. Therefore if $A \in f(s)$ then $B \in f(s)$.

Goldblatt-Thomason Theorem

For (2), we first prove that

$\prod_U(\mathcal{G}, V'), s \equiv_{\mathtt{ML}^+} \mathfrak{ue}(\mathcal{F}, V), u \iff f(s) = u$

$\Rightarrow$: Suppose $s \equiv_{\mathtt{ML}^+} u$. It is clear that for any p_A:

$s \vDash p_A \iff u \vDash p_A \overset{\text{property of } \mathfrak{ue}}{\iff} V(p_A) \in u \iff A \in u$ therefore by definition of f we have $f(s) = u$.

$\Leftarrow$: Suppose $f(s) = u$. For any $\varphi \in \mathtt{ML}^+$:

$u \vDash \varphi \iff V(\varphi) \in u \overset{f(s)=u}{\iff} s \vDash p_{V(\varphi)} \overset{\mathcal{F}, V \vDash p_{V(\varphi)} \leftrightarrow \varphi}{\iff} s \vDash \varphi$.

Recall that modally equivalence coincides with bisimulation w.r.t m-saturated models. Therefore f essentially denotes a bisimulation between $\mathfrak{ue}(\mathcal{F}, V)$ and $\prod_U(\mathcal{G}, V')$: we can define a bisimulation relation Z as $(s, u) \in Z \iff f(s) = u$. Clearly f is a bounded morphism at the frame level from $\prod_U \mathcal{G}$ to $\mathfrak{ue}(\mathcal{F})$.

Goldblatt-Thomason Theorem

For (3), we NTS that for any u there is an s such that $f(s) = u$, namely, for any u there is an s such that $s \vDash \{p_A \mid A \in u\}$. Note that $\prod_U(\mathcal{G}, V')$ is countably saturated, thus if a set of formulae is finitely satisfiable in $\prod_U(\mathcal{G}, V')$ then it is satisfiable in $\prod_U(\mathcal{G}, V')$. Now we only need to show that $\{p_A \mid A \in u\}$ is finitely satisfiable in the ultrapower. Since u is an ultrafilter then u has the finite intersection property. It means that $\mathcal{F}, V \vDash (p_{A_1} \wedge \cdots \wedge p_{A_n}) \leftrightarrow p_B$ for some $B \neq \emptyset$. Thus $\prod_U(\mathcal{G}, V') \vDash (p_{A_1} \wedge \cdots \wedge p_{A_n}) \leftrightarrow p_B$. We only NTS p_B is satisfiable in $\prod_U(\mathcal{G}, V')$. First note that p_B is clearly satisfiable in $\mathcal{F}, V$ which is assumed to be w-generated. Therefore $\Diamond^n p_B$ holds at w. Thus $\mathcal{G}, V', v$ satisfies $\Diamond^n p_B$. Then p_B is clearly satisfiable in $\prod_U(\mathcal{G}, V')$.

You can also do without the p_A.

Goldblatt-Thomason Theorem

Theorem (Goldblatt-Thomason Theorem)

A first-order definable class $\mathbb{K}$ of frames is definable by a set of modal formulas if and only if it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions.

Is there a direct, more modal characterizaiton?

Adding closure of ultraproduct or ultrafilter union does not work.

Summary

Important concepts: local/global frame definability, relative definability, monadic second-order logic, standard translation to MSO, local/global first-order correspondent, frame construction methods: disjoint union, generated subframe, bounded morphic image, ultrafilter extension, positive/negative occurrence, uniform formula, positive/negative formulas,upward/downward monotonicity, Sahlqvist implication, Sahlqvist formula, McKinsey Formula, restricted quantifier, inherent universality, Kracht formula, Jankov-Fine formula, finite transitive frames

Important results: Sahlqvist theorem, Kracht theorem, Chagrova's theorem, Goldblatt-Thomason theorem