Advanced Modal Logic XVII

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April 18th, 2023

Advanced Modal Logic (2023 Spring)

Kracht Fragment

Kracht Fragment

The first-order “Sahlqvist fragment”

$p \to \Diamond p$	$\exists y(xRy \wedge y = x)$
$\Box p \to \Box\Box p$	$\forall y(xRy \to \forall z(yRz \to xRz))$
$\Diamond p \to \Diamond\Diamond p$	$\forall y(xRy \to \exists t(xRt \wedge \exists z(tRz \wedge z = y)))$
$(p \wedge \Diamond\Diamond p) \to \Diamond p$	$\forall y \forall z((xRy \wedge yRz) \to \exists t(xRt \wedge (t = x \vee t = z)))$ Guar
$\Diamond\Box p \to \Box\Diamond p$	$\forall y(xRy \to \forall s(xRs \to \exists t(sRt \wedge yRt)))$
$(p \wedge \Diamond\neg p) \to \Diamond p$	$\forall y(xRy \to y = x \vee (\exists z(xRz \wedge z = x)))$
$\Box(p \to \Diamond p)$	$\forall y(xRy \to \exists z(yRz \wedge z = y))$

quantifiers: $\exists y \rhd x$, $\forall y \rhd x$. In general, modal logic is a type of (decidable) *guarded fragments* of FOL.

Guarded quantifiers act like quantifiers:

$$\neg\exists y \rhd x \neg\alpha \leftrightarrow \forall y \rhd x\alpha$$

However, there are also crucial differences, and the following is invalid (e.g., take $Q^r = \forall x \rhd y$, $\heartsuit = \wedge$ and $\gamma = \bot$)

$$(Q^r x\beta)\heartsuit\gamma \leftrightarrow Q^r x(\beta\heartsuit\gamma)$$ (3.20 in the blue book is invalid)

$p \to \Diamond p$	$\exists y \rhd x(y = x)$
$\Box p \to \Box\Box p$	$\forall y \rhd x \forall z \rhd y(xRz)$
$\Diamond p \to \Diamond\Diamond p$	$\forall y \rhd x \exists t \rhd x \exists z \rhd t(z = y)$
$(p \wedge \Diamond\Diamond p) \to \Diamond p$	$\forall y \rhd x \forall z \rhd y \exists t \rhd x(t = x \vee t = z)$
$\Diamond\Box p \to \Box\Diamond p$	$\forall y \rhd x \forall s \rhd x \exists t \rhd s(yRt)$
$(p \wedge \Diamond\neg p) \to \Diamond p$	$\forall y \rhd x(y = x \vee (\exists z \rhd x(z = x)))$
$\Box(p \to \Diamond p)$	$\forall y \rhd x \exists z \rhd y(z = y)$

In the “converted atomic formulas” at least one variable is free or universally bounded.

Kracht Formula

A *clean* formula is a formula in which no variable occurs both free and bound, and no two distinct (occurrences of) quantifiers bind the same variable.

A Kracht formula (as defined in the blue book) is a clean formula (with one free variable x_0) built from atomic formulas ($u = v, vRu$ and $u \neq u$) by using $\wedge, \vee$ and restricted (guarded) quantifiers $\forall x \triangleright u$ and $\exists x \triangleright u$ such that at least one variable x in any non-trivial atomic formulas (excl. $u = u$ and $u \neq u$) is *inherently universal*: x is either free or x is bound by $\forall x \triangleright u$ which is not in the scope of any existential quantifier.

Claim: Every Sahlqvist formula corresponds to a Kracht formula. Is it really true? Consider the formula $\Box\Box p \to \Diamond p$.

We need to allow $xR^n y$ as "atomic formula" as well!

Kracht Algorithm

The core of Sahlqvist algorithm turns a Sahlqvist implication

$ST_x(A \to \psi) = \forall P_1 \ldots \forall P_n \forall x_1 \ldots \forall x_m (\mathtt{REL} \wedge \mathtt{BOX\text{-}AT} \to \neg \mathtt{NEG} \vee ST_x(\psi))$

into:

$$\forall x_1 \ldots \forall x_m (\mathtt{REL} \to \alpha)$$

A Kracht formula can be rewritten into a Sahlqvist formula by the following three general steps as on the blue book:

1. massage it into a prenex form (not always possible!)
2. rewrite POS and recover BOX-AT
3. absorb the restricted quantifiers by $\Box, \Diamond$

Claim: Every Kracht formula is equivalent to a Sahlqvist formula (not just Sahlqvist implication). The proof in the blue book has many problems and cannot be fixed. We will only explain the basic ideas by exmpales here.

Kracht Formula

Let x_i be an inherently universal variable.

Atom	POS	VAL	BOX-AT
$u = x_i$	$ST_u(p_i)$	$P_i u := \ u = x_i$	$ST_{x_i}(p_i) \rightarrow$
$u \neq u$	$ST_u(\bot)$		
$u = u$	$ST_u(\top)$		
$x_i R u$	$ST_u(q_i)$	$Q_i u := \ x_i R u$	$ST_{x_i}(\Box q_i) \rightarrow$
$u R x_i$	$ST_u(\Diamond p_i)$	$P_i u := \ u = x_i$	$ST_{x_i}(p_i) \rightarrow$

For the last clause:

$ST_u(\Diamond p_i)[P_i u := \ u = x_i]$

$= \exists z(uRz \wedge Pz)[P_i u := \ u = x_i] = \exists z(uRz \wedge z = x_i) \leftrightarrow uRx_i$

What about $x_i R^n u$ and $u R^n x_i$? We need $\Box^n$ and $\Diamond^n$. correspondingly.

Kracht Formula

1. rewrite POS and recover BOX-AT
2. absorb the restricted quantifiers by $\Box, \Diamond$

Example ($x_0 R x_0$, the underlined variable is the selected inherently universal variable)

Atom	POS	VAL	BOX-AT
$\underline{x_0} R x_0$	$ST_{x_0}(q)$	$Qu := x_0 R u$	$ST_{x_0}(\Box q) \rightarrow$

$$\forall Q(ST_{x_0}(\Box q) \rightarrow ST_{x_0}(q)) = \forall Q(ST_{x_0}(\Box q \rightarrow q))$$

Kracht Formula

Example ($\exists x_1 \rhd x_0(x_1 R x_0)$)

Atom	POS	VAL	BOX-AT
$x_1 R \underline{x_0}$	$ST_{x_1}(\Diamond p)$	$Pu := \; u = x_0$	$ST_{x_0}(p) \rightarrow$

$\forall P(ST_{x_0}(p) \rightarrow \exists x_1 \rhd x_0(ST_{x_1}(\Diamond p))) = \forall P(ST_{x_0}(p \rightarrow \Diamond\Diamond p))$

Kracht Formula

Example ($x_0 R x_0$)

Atom	POS	VAL	BOX-AT
$\underline{x_0} R x_0$	$ST_{x_0}(q)$	$Qu := \ x_0 Ru$	$ST_{x_0}(\Box q) \rightarrow$
$x_0 R \underline{x_0}$	$ST_{x_0}(\Diamond p)$	$Pu := \ u = x_0$	$ST_{x_0}(p) \rightarrow$

$$\forall Q(ST_{x_0}(\Box q) \rightarrow ST_{x_0}(q)) = \forall Q(ST_{x_0}(\Box q \rightarrow q))$$
$$\forall P(ST_{x_0}(p) \rightarrow ST_{x_0}(\Diamond p)) = \forall P(ST_{x_0}(p \rightarrow \Diamond p))$$

Kracht Formula

Example ($\forall x_1 \rhd x_0\ \forall x_2 \rhd x_1\ x_0Rx_2$)

Atom	POS	VAL	BOX-AT
$\underline{x_0}Rx_2$	$ST_{x_2}(q)$	$Qu := \ x_0Ru$	$ST_{x_0}(\Box q) \rightarrow$
$x_0R\underline{x_2}$	$ST_{x_0}(\Diamond p)$	$Pu := \ u = x_2$	$ST_{x_2}(p) \rightarrow$

$\forall Q(\forall x_1 \rhd x_0 \forall x_2 \rhd x_1(ST_{x_0}(\Box q) \rightarrow ST_{x_2}(q)))$

$\forall Q(\neg\exists x_1 \rhd x_0 \exists x_2 \rhd x_1 \neg(ST_{x_0}(\Box q) \rightarrow ST_{x_2}(q)))$

$\forall Q(\neg\exists x_1 \rhd x_0 \exists x_2 \rhd x_1(ST_{x_0}(\Box q) \wedge ST_{x_2}(\neg q)))$

$\forall Q(\neg\exists x_1 \rhd x_0(ST_{x_0}(\Box q) \wedge ST_{x_1}(\Diamond\neg q)))$

$\forall Q\neg(ST_{x_0}(\Box q) \wedge ST_{x_0}(\Diamond\Diamond\neg q))$

$\forall Q(ST_{x_0}(\Box q \rightarrow \Box\Box q))$

$\exists y \rhd x(\alpha(y) \wedge \beta) \leftrightarrow \exists y \rhd x\alpha(y) \wedge \beta$ is valid. If y does not occur free in β.

What about turning a Sahlqvist formula into a Kracht formula and back to a Sahlqvist formula?

Kracht Formula

Example ($\forall x_1 \rhd x_0\ \forall x_2 \rhd x_1\ x_0 R x_2$)

Atom	POS	VAL	BOX-AT
$\underline{x_0} R x_2$	$ST_{x_2}(q)$	$Qu := \ x_0 R u$	$ST_{x_0}(\Box q) \rightarrow$
$x_0 R \underline{x_2}$	$ST_{x_0}(\Diamond p)$	$Pu := \ u = x_2$	$ST_{x_2}(p) \rightarrow$

$\forall P \forall x_1 \rhd x_0 \forall x_2 \rhd x_1 (ST_{x_2}(p) \rightarrow ST_{x_0}(\Diamond p))$

$\forall P \neg \exists x_1 \rhd x_0 \exists x_2 \rhd x_1 (ST_{x_2}(p) \wedge ST_{x_0}(\neg \Diamond p))$

$\forall P \neg (ST_{x_0}(\Diamond \Diamond p) \wedge ST_{x_0}(\neg \Diamond p))$

$\forall P (ST_{x_0}(\Diamond \Diamond p \rightarrow \Diamond p))$

Kracht Formula

Example ($\forall x_1 \rhd x_0\ \forall x_2 \rhd x_1\ (x_0 R x_0 \vee x_0 R x_2)$)

Atom	POS	VAL	BOX-AT
$\underline{x_0} R x_2$	$ST_{x_2}(q)$	$Qu := \ x_0 R u$	$ST_{x_0}(\Box q) \rightarrow$
$\underline{x_0} R x_0$	$ST_{x_0}(q)$	$Qu := \ x_0 R u$	$ST_{x_0}(\Box q) \rightarrow$

$\forall Q \forall x_1 \rhd x_0 \forall x_2 \rhd x_1 (ST_{x_0}(\Box q) \rightarrow (ST_{x_0}(q) \vee ST_{x_2}(q)))$

$\forall Q \neg \exists x_1 \rhd x_0 \exists x_2 \rhd x_1 (ST_{x_0}(\Box q) \wedge (\neg ST_{x_0}(q) \wedge \neg ST_{x_2}(q)))$

$\forall Q \neg \exists x_1 \rhd x_0 \exists x_2 \rhd x_1 (ST_{x_0}(\Box q \wedge \neg q) \wedge ST_{x_2}(\neg q))$

$\forall Q \neg \exists x_1 \rhd x_0 (ST_{x_0}(\Box q \wedge \neg q) \wedge ST_{x_1}(\Diamond \neg q))$

$\forall Q \neg (ST_{x_0}(\Box q \wedge \neg q) \wedge ST_{x_0}(\Diamond \Diamond \neg q)$

$\forall Q (ST_{x_0}((\neg q \wedge \Diamond \Diamond \neg q) \rightarrow \Diamond \neg q)$

Kracht Formula

Example ($\forall x_1 \rhd x_0\ \forall x_2 \rhd x_0\ \exists y \rhd x_1 (x_2 R y)$)

Atom	POS	VAL	BOX-AT
$\underline{x_2} Ry$	$ST_y(q)$	$Qu := \ x_2 Ru$	$ST_{x_2}(\Box q) \rightarrow$

$\forall Q \forall x_1 \rhd x_0 \forall x_2 \rhd x_0 (ST_{x_2}(\Box q) \rightarrow \exists y \rhd x_1 (ST_y(q)))$

$\forall Q \forall x_1 \rhd x_0 \forall x_2 \rhd x_0 (ST_{x_2}(\Box q) \rightarrow ST_{x_1}(\Diamond q))$

$\forall Q \neg \exists x_1 \rhd x_0 \exists x_2 \rhd x_0 (ST_{x_2}(\Box q) \wedge ST_{x_1}(\neg \Diamond q))$

$\forall Q \neg (ST_{x_0}(\Diamond \Box q) \wedge ST_{x_0}(\Diamond \neg \Diamond q))$

$\forall Q ST_{x_0}(\Diamond \Box q \rightarrow \Box \Diamond q)$

Kracht Formula

Example ($\forall x_1 \rhd x_0\ (x_1 = x_0 \vee x_0 R x_0)$)

Atom	POS	VAL	BOX-AT
$x_1 = \underline{x_0}$	$ST_{x_1}(p)$	$Pu := \ u = x_0$	$ST_{x_0}(p) \rightarrow$
$\underline{x_0} R x_0$	$ST_{x_0}(q)$	$Qu := \ x_0 R u$	$ST_{x_0}(\Box q) \rightarrow$

$$\forall Q \forall P \forall x_1 \rhd x_0 (ST_{x_0}(p \wedge \Box q) \rightarrow ST_{x_1}(p) \vee ST_{x_0}(q))$$

$$\forall Q \forall P \neg \exists x_1 \rhd x_0 (ST_{x_0}(p \wedge \Box q) \wedge ST_{x_1}(\neg p) \wedge ST_{x_0}(\neg q))$$

$$\forall Q \forall P \neg (ST_{x_0}(p \wedge \neg q \wedge \Box q \wedge \Diamond \neg p))$$

$$\forall Q \forall P\ ST_{x_0}(p \wedge \neg q \wedge \Box q \rightarrow \Box p)$$

$$\forall Q \forall P\ ST_{x_0}(p \wedge \Box q \rightarrow \Box p \vee q)$$

Kracht Formula

Example $(\forall x_1 \rhd x_0\ (x_1 = x_0 \lor x_0 R x_0))$

Atom	POS	VAL	BOX-AT
$x_1 = \underline{x_0}$	$ST_{x_1}(p)$	$Pu := \ u = x_0$	$ST_{x_0}(p) \rightarrow$
$x_0 R \underline{x_0}$	$ST_{x_0}(\Diamond p)$	$Pu := \ u = x_0$	$ST_{x_0}(p) \rightarrow$

$\forall P \forall x_1 \rhd x_0 (ST_{x_0}(p) \rightarrow ST_{x_1}(p) \lor ST_{x_0}(\Diamond p))$

$\forall P \neg \exists x_1 \rhd x_0 (ST_{x_0}(p) \land ST_{x_1}(\neg p) \land ST_{x_0}(\neg \Diamond p))$

$\forall P \neg (ST_{x_0}(p) \land ST_{x_0}(\Diamond \neg p) \land ST_{x_0}(\neg \Diamond p))$

$\forall P\ ST_{x_0}(p \land \Diamond \neg p \rightarrow \Diamond p)$

$\forall P\ ST_{x_0}(p \rightarrow \Box p \lor \Diamond p)$

Kracht's approach

For a correct proof, we need to generalize the definability to first-order formulas with multiple varables:

$\alpha(x_0, \ldots, x_n)$ is modally definable iff there are modal formulas $\varphi_0, \ldots, \varphi_n$ such that for all $\mathcal{F}$, for any worlds $w_0, \ldots, w_n$:

$\mathcal{F} \Vdash \alpha(x_0, \ldots, x_n)[w_0, \ldots, w_n]$ iff for any valuation V, there exists $i \leq n$ such that $\mathcal{F}, V, w_i \vDash \varphi_i$. E.g., $x_1 R x_2$ is definable by $\Diamond\neg p, p$.

Theorem (Kracht (1999))

If $\alpha(x_0)$ is obtained from definable formulas using conjunction, disjunction and restricted universal quantification, then $\alpha(x_0)$ is modally definable.

We need to show Kracht formulas are obtained from definable formuals as the above.