Advanced Modal Logic XVI

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April 12th, 2023

Advanced Modal Logic (2023 Spring)

Sahlqvist Formulas (cont.)

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$\varphi \to POS$: a summary of the cases

- $\varphi = p$ (e.g., $p \to \Diamond p$)
- $\varphi = \Box \dots \Box p$ (e.g., $\Box p \to \Box\Box p$)
- $\varphi = \Diamond\varphi$ (e.g., $\Diamond p \to \Diamond\Diamond p$): $\exists y \alpha(y) \to \beta \quad \leftrightarrow \quad \forall y(\alpha(y) \to \beta)$
- $\varphi = \psi \wedge \Diamond\psi'$ (e.g., $(p \wedge \Diamond p) \to \Diamond p$):$u = x \vee u = z$, $\alpha \wedge (\vee)\exists y \beta(y) \leftrightarrow \exists y(\alpha \wedge (\vee)\beta(y))$
- $\varphi = \psi \vee \psi'$ (e.g., $(p \vee \Box p) \to \Diamond p$): $(\alpha \vee \beta) \to \gamma \quad \leftrightarrow \quad (\alpha \to \gamma \wedge \beta \to \gamma)$, $\forall \dots (\alpha \wedge \beta) \quad \leftrightarrow \quad (\forall \dots \alpha \wedge \forall \dots \beta)$
- $\varphi = \Diamond\Box\psi$ (e.g., $\Diamond\Box p \to \Box\Diamond p$)
- $\varphi = \psi \wedge \texttt{NEG}$ (e.g., $(p \wedge \Diamond\neg p) \to \Diamond p$): $(\alpha \wedge \beta) \to \gamma \quad \leftrightarrow \quad \alpha \to (\neg\beta \vee \gamma)$.

Sahlqvist Implication

Definition (Sahlqvist implication for the basic similarity type)

A Sahlqvist antecedent is a formula built up from $\bot$, $\top$, $\Box \dots \Box p$, basic proposition letters and negative formulas, using $\land$, $\lor$ and $\Diamond$. A Sahlqvist implication $\varphi \to \psi$ is an implication in which ψ is positive and φ is a Sahlqvist antecedent.

Take those building blocks $\bot$, $\top$, $\Box ... \Box p$ and negative formulas as if they were atomic formulas.

Sahlqvist Implication

Sahlqvist-van Benthem Algorithm:

1. renaming
2. pull out all the diamonds
3. disj nf, distributing disjunctions
4. rewrite the translation into a conjunctive form of:

$$\forall P_1 \ldots \forall P_n \forall x_1 \ldots \forall x_m (\mathtt{REL} \wedge \mathtt{BOX\text{-}AT} \wedge \mathtt{NEG} \rightarrow ST_x(\psi))$$

5. transfer the NEG part:

$$\forall P_1 \ldots \forall P_n \forall x_1 \ldots \forall x_m (\mathtt{REL} \wedge \mathtt{BOX\text{-}AT} \rightarrow \neg\mathtt{NEG} \vee ST_x(\psi))$$

6. read off the minimal valuation and eliminate $\forall P_1 \ldots \forall P_n$ (and BOX-AT).

Sahlqvist formula

Definition (for the basic similarity type)

A Sahlqvist formula is a formula that is built up from Sahlqvist implications by freely applying boxes and conjunctions, and by applying disjunctions only between formulas that do not share any proposition letters.

Sahlqvist formula

Lemma

- *φ and $\alpha(x)$ are local correspondents, then so are $\Box\varphi$ and $\forall y(Rxy \to \alpha(y))$.*
- *If φ (locally) corresponds to α and ψ (locally) corresponds to β then $\varphi \wedge \psi$ (locally) corresponds to $\alpha \wedge \beta$.*
- *If φ locally corresponds to α, ψ locally corresponds to β, and φ and ψ have no propositional letters in common, then $\varphi \vee \psi$ locally corresponds to $\alpha \vee \beta$.*

Theorem

Given a Sahlqvist formula, we can compute its local first-order correspondent.

Sahlqvist formula

Note that the following is **false**:

φ and $\alpha(x)$ are local correspondents, then so are $\Diamond\varphi$ and $\exists y(Rxy \wedge \alpha(y))$.

To see this, note that:

$$\begin{aligned} &\mathcal{F}, w \vDash \Diamond\varphi \\ \Leftrightarrow\ &\text{For any } V \text{ there exists } v \text{ such that } wRv : \mathcal{F}, V, v \vDash \varphi \\ \not\Leftrightarrow\ &\text{There exists } v \text{ such that } wRv \text{ and for any } V : \mathcal{F}, V, v \vDash \varphi \end{aligned}$$

A counter example is $\Diamond(p \to \Diamond p)$ (it is equivalent to a Sahlqvist formula): although $p \to \Diamond p$ locally corresponds to xRx but $\Diamond(p \to \Diamond p)$ does not locally correspond to $\exists y(xRy \wedge yRy)$.

Extending the Sahlqvist fragment

What about $\Box\Diamond p \to \Diamond\Box p$ and $\Box(p \vee q) \to \Diamond(\Box p \vee \Box q)$?

Note that there are many incomparable minimal valuations which can make the antecedents in the above formulas true. We need to “universally quantify” over such minimal valuations in terms of first-order formulas. Recall that for $\Diamond p$ we can do so by pulling out the $\Diamond$ and enforce $\forall y\, ((xRy \wedge Py) \to \beta)$ (Intuitively we can talk about each minimal valuation $\{y\}$ by fixing a y first. Thus $\forall y$ quantifies over all these valuations). Here we cannot use this trick since the minimal valuations are in the form of *choice functions*: for each successor, select a world (to make p true) which are inherently second order objects.

What about other similarity types?

$$\begin{array}{lcl} \mathcal{M}, w \models \triangle(\varphi_1 \dots \varphi_{\rho(\triangle)}) & \Leftrightarrow & \exists w_1 \dots w_{\rho(\triangle)} : \langle w, w_1, \dots, w_{\rho(\triangle)} \rangle \in R \\ & & \text{and } \forall i \in [1, \rho(\triangle)] : \mathcal{M}, w_i \models \varphi_i \\ \mathcal{M}, w \models \nabla(\varphi_1 \dots \varphi_{\rho(\nabla)}) & \Leftrightarrow & \forall w_1 \dots w_{\rho(\nabla)} : \langle w, w_1, \dots, w_{\rho(\nabla)} \rangle \in R \\ & & \implies \exists i \in [1, \rho(\nabla)] : \mathcal{M}, w_i \models \varphi_i \end{array}$$

Can we use Sahlqvist algorithm for $\triangle(p, p) \to p$? Yes! What about $\nabla(p, p) \to p$? No! (same reason as in the case of $\Box(\vee)$).

Thus **not** every result about basic modal logic can be generalized to arbitrary similarity types by replacing $\Box$ by ∇!

Definition (Sahlqvist implication for arbitrary similarity type)

A Sahlqvist antecedent is a formula built up from $\bot$, $\top$, $\Box_1 \dots \Box_n p$ ($n \geq 0$), and negative formulas, using $\wedge$, $\vee$, $\Diamond$ and $\triangle$. A Sahlqvist implication $\varphi \to \psi$ is an implication in which ψ is positive and φ is a Sahlqvist antecedent.

A Limitative result

There are modal formulas which do not have local first-order correspondents (thus not Sahlqvist) but do have global correspondents. e.g, $M \wedge 4 : (\Box\Diamond p \to \Diamond\Box p) \wedge (\Diamond\Diamond p \to \Diamond p)$.

There are also modal formulas which have local first-order correspondents (and also the global ones) but are not equivalent to any Sahlqvist formula: $\Box M \wedge 4$. Thus Sahlqvist fragment does not cover all the modal formulas which have (local) FO-correspondents.

Can we extend it such that it is still decidable and it covers all the FO-translatable modal formulas? NO!

Theorem (Chagrova's Theorem)

It is undecidable whether an arbitrary basic modal formula has a first-order correspondent.