



Advanced Modal Logic XV

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Advanced Modal Logic (2023 Spring)

Sahlqvist Formulas

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Towards the Sahlqvist fragment

Which modal formulas have first-order correspondents?

Answer 1: if φ is **closed** (it does not mention any proposition letter) then its second-order translation is without any unary predicate thus it is a first-order formula.

Answer 2: if φ has a **local** correspondent $\alpha(x)$ (i.e., for all \mathcal{F}, w : $\mathcal{F}, w \models \varphi \iff \mathcal{F} \Vdash \alpha(x)[w]$) then it has a (global) correspondent. For example, $\Box p \rightarrow p$ has a local correspondent Rxx and it has the first-order correspondent $\forall x Rxx$.

Answer 3: if φ has some nice property then we can **eliminate** the universal quantifiers over valuations by considering the “representatives” of valuations only.

Towards the Sahlqvist fragment

Definition (Positive occurrences)

An occurrence of a proposition letter p is a **positive** occurrence if it is in the scope of an even number of negation signs; it is a **negative** occurrence if it is in the scope of an odd number of negation signs. A modal formula φ is **positive in p** (negative in p) if all occurrences of p are positive (negative). A formula is called positive (negative) if it is positive (negative) in all proposition letters occurring in it.

For example, q occurs positively in $\diamond(p \rightarrow \Box q)$ while p occurs negatively in it (\rightarrow is not a primitive connective). Standard translation preserves positive/negative occurrences of proposition letters. Why?

Uniform formulas

Definition (Monotonicity)

A modal formula is *upward monotone* in p if its truth is preserved under extensions of $V(p)$. Likewise, a formula is *downward monotone* in p if its truth is preserved under shrinkings of $V(p)$.

In the other words, if V and V' only differ in p :

φ is upward monotone if for any \mathcal{F} and any V and V' over \mathcal{F} :

$$\text{Ceteris paribus, } V(p) \subseteq V'(p) \implies \llbracket \varphi \rrbracket^{\mathcal{F}, V} \subseteq \llbracket \varphi \rrbracket^{\mathcal{F}, V'}$$

φ is downward monotone if for any \mathcal{F} and V and any V' over \mathcal{F} :

$$\text{Ceteris paribus, } V'(p) \subseteq V(p) \implies \llbracket \varphi \rrbracket^{\mathcal{F}, V} \subseteq \llbracket \varphi \rrbracket^{\mathcal{F}, V'}$$

Uniform formulas

It is not hard to show that:

Proposition

For any modal formula φ :

- (1) If φ is positive in p , then it is upward monotone in p ,
- (2) if φ is negative in p , then it is downward monotone in p .

Proof.

We need a *simultaneous induction* here: prove (1) and (2) together. Recall the result about the modal preservation of ultrafilter extensions: the stronger claim is easier to prove since you can use stronger induction hypothesis. \square

Uniform formulas

Definition

A modal formula is *uniform* in p if all the occurrences of p are positive or all the occurrences of p are negative. A formula is uniform if it is uniform in all proposition letters.

Theorem

Every uniform modal formula has a first-order correspondent.

Proof.

(1) Eliminating all the $\forall P$ in the (local) second-order standard translation and (2) Replacing Px by $\neg x = x$ if φ is positive in p and replacing Px by $x = x$ if φ is negative in p . (3) Make it global. □

$P(\cdot)$ can be defined by λ expressions (e.g., $\lambda u.u = u$). Examples:

$\diamond\diamond\square p, \neg p \rightarrow \diamond p\dots$

Towards the Sahlqvist fragment

What about formulas that are not uniform ($A \rightarrow \text{POS}$)?

Example ($p \rightarrow \diamond p$)

Let us try to eliminate the quantifiers in the *local* MSO translation: $\forall P(Px \rightarrow \exists y(xRy \wedge Py))$

Note that the consequent is positive in p , we just need to find the *minimal valuation* of P which can make the antecedent true. Why? To see this, first note that for all \mathcal{F}, w :

$$\mathcal{F} \Vdash \forall P(Px \rightarrow \exists y(xRy \wedge Py))[w] \implies \mathcal{F} \Vdash (Px \rightarrow \exists y(xRy \wedge Py))[X, w]$$

where X is the minimal valuation for P which makes Px true. For the converse, suppose $\mathcal{F} \Vdash Px \rightarrow \exists y(xRy \wedge Py)[X, w]$. Now consider an arbitrary $Y \subseteq W_{\mathcal{F}}$, if $\mathcal{F} \Vdash Px[Y, w]$ then $X \subseteq Y$ (since X is minimal valuation making Px true). (to be continued.)

Towards the Sahlqvist fragment

Example ($p \rightarrow \Diamond p$ cont.)

Since the consequent is positive in p thus by the upward monotonicity $\mathcal{F} \Vdash \exists y(xRy \wedge Py)[Y, w]$, thus

$\mathcal{F} \Vdash (Px \rightarrow \exists y(xRy \wedge Py))[Y, w]$ for arbitrary Y . Therefore

$\mathcal{F} \Vdash \forall P(Px \rightarrow \exists y(xRy \wedge Py))[w]$.

Let $Pu := \mathbf{u=x}$, we now eliminate $\forall P$ and the antecedent: $\exists y(xRy \wedge \mathbf{y=x})$ which is (locally) equivalent to xRx .

Example ($\Box p \rightarrow \Box \Box p$: antecedent with \Box)

$\forall P(\forall y(xRy \rightarrow Py) \rightarrow \forall y(xRy \rightarrow \forall z(yRz \rightarrow Pz)))$

Let $Pu := \mathbf{xRu}$, we now eliminate $\forall P$ and the antecedent:

$\forall y(xRy \rightarrow \forall z(yRz \rightarrow \mathbf{xRz}))$.

Towards the Sahlqvist fragment

Example ($\diamond p \rightarrow \diamond\diamond p$: antecedent with \diamond)

$\forall P(\exists y(xRy \wedge Py) \rightarrow \exists y(xRy \wedge \exists z(yRz \wedge Pz)))$

Now what is the minimal valuation satisfying the antecedent?

We need to somehow pick up an *arbitrary* y and make p true there.

Note that (if y does not appear freely in β):

$$(\exists y\alpha(y)) \rightarrow \beta \quad \leftrightarrow \quad \forall y(\alpha(y) \rightarrow \beta).$$

We can then pull out the existential quantifier:

$\forall P\forall y((xRy \wedge Py) \rightarrow \exists y(Rxy \wedge \exists z(yRz \wedge Pz)))$

Let $Pu := \mathbf{u=y}$, we now eliminate $\forall P$:

$\forall y(xRy \rightarrow \exists y(xRy \wedge \exists z(yRz \wedge \mathbf{z=y})))$ **Ouch!**

Towards the Sahlqvist fragment

Example ($\diamond p \rightarrow \diamond\diamond p$: antecedent with \diamond)

$\forall P(\exists y(xRy \wedge Py) \rightarrow \exists t(Rxt \wedge \exists z(tRz \wedge Pz)))$

Now what is the minimal valuation satisfying the antecedent?

We need to somehow pick up an *arbitrary* y and make p true there. Note that (if y does not appear in β)

$$(\exists y \alpha(y)) \rightarrow \beta \quad \leftrightarrow \quad \forall y(\alpha(y) \rightarrow \beta).$$

We can then pull out the existential quantifier:

$\forall P\forall y((xRy \wedge Py) \rightarrow \exists t(xRy \wedge \exists z(tRz \wedge Pz)))$

Let $Pu := \mathbf{u=y}$, we now eliminate $\forall P$:

$\forall y(xRy \rightarrow \exists t(xRt \wedge \exists z(tRz \wedge \mathbf{z=y})))$

Towards the Sahlqvist fragment

Example $((p \wedge \Diamond\Diamond p) \rightarrow \Diamond p)$: antecedent with \wedge

$\forall P((Px \wedge \exists y(xRy \wedge (\exists z(yRz \wedge Pz)))) \rightarrow \exists t(xRt \wedge Pt))$ By using the following tautology (y does not appear freely in α):

$$\alpha \wedge \exists y\beta(y) \leftrightarrow \exists y(\alpha \wedge \beta(y))$$

we move the existential quantifiers:

$$\forall P(\exists y\exists z(Px \wedge xRy \wedge yRz \wedge Pz) \rightarrow \exists t(xRt \wedge Pt))$$

Pulling out \exists : $\forall P\forall y\forall z((Px \wedge xRy \wedge yRz \wedge Pz) \rightarrow \exists t(xRt \wedge Pt))$

Let $Pu := \mathbf{u=x} \vee \mathbf{u=z}$, we now eliminate $\forall P$:

$\forall y\forall z(xRy \wedge yRz) \rightarrow \exists t(xRt \wedge (\mathbf{t=x} \vee \mathbf{t=z}))$ which can be simplified as: $\forall y\forall z(xRy \wedge yRz) \rightarrow (xRx \vee xRz)$.

Towards the Sahlqvist fragment

Example $((p \vee \Box p) \rightarrow \Diamond p)$: antecedent with \vee

$\forall P((Px \vee \forall y(xRy \rightarrow Py)) \rightarrow (\exists z(xRz \wedge Pz)))$ By using the following tautology:

$$(\alpha \vee \beta) \rightarrow \gamma \quad \leftrightarrow \quad ((\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma))$$

we turn the disjunction into a conjunction:

$\forall P((Px \rightarrow (\exists z(xRz \wedge Pz))) \wedge (\forall y(xRy \rightarrow Py) \rightarrow (\exists z(xRz \wedge Pz))))$. By using the following equality:

$$\forall \dots (\alpha \wedge \beta) \quad \leftrightarrow \quad (\forall \dots \alpha \wedge \forall \dots \beta)$$

$\forall P((Px \rightarrow (\exists z(xRz \wedge Pz))) \wedge \forall P(\forall y(xRy \rightarrow Py) \rightarrow (\exists z(xRz \wedge Pz))))$.

We can continue with each of the two conjuncts.

Example (Geach Axiom: $\diamond\Box p \rightarrow \Box\diamond p$: \diamond over \Box)

$\forall P(\exists y(xRy \wedge \forall z(yRz \rightarrow Pz)) \rightarrow \forall s(xRs \rightarrow \exists t(sRt \wedge Pt)))$

We can then pull out the existential quantifier:

$\forall P\forall y(xRy \wedge \forall z(yRz \rightarrow Pz)) \rightarrow \forall s(xRs \rightarrow \exists t(sRt \wedge Pt))$

Let $Pu := \mathbf{yRu}$, we now eliminate $\forall P$:

$\forall y(xRy \rightarrow \forall s(xRs \rightarrow \exists t(sRt \wedge \mathbf{yRt})))$

Towards the Sahlqvist fragment

Example $((p \wedge \diamond\neg p) \rightarrow \diamond p$: antecedent with negative formulas)

$\forall P(Px \wedge \exists y(xRy \wedge \neg Py) \rightarrow \exists z(xRz \wedge Pz))$

By using the following tautology (y does not appear in α):

$$\alpha \wedge \exists y\beta(y) \leftrightarrow \exists y(\alpha \wedge \beta(y))$$

we push the existential quantifier out of conjunctions:

$\forall P(\exists y(Px \wedge xRy \wedge \neg Py) \rightarrow \exists z(xRz \wedge Pz))$

Pulling out the existential quantifier:

$\forall P\forall y((Px \wedge xRy \wedge \neg Py) \rightarrow \exists z(xRz \wedge Pz))$

Towards the Sahlqvist fragment

Example $((p \wedge \Diamond \neg p) \rightarrow \Diamond p)$: cont.)

Pulling out the existential quantifier:

$$\forall P \forall y ((Px \wedge xRy \wedge \neg Py) \rightarrow \exists z (xRz \wedge Pz))$$

By using the following tautology:

$$(\alpha \wedge \beta) \rightarrow \gamma \quad \leftrightarrow \quad \alpha \rightarrow (\neg \beta \vee \gamma)$$

(note that we can also massage the modal formula at the first place) we move the negative part to the consequent:

$$\forall P \forall y ((Px \wedge xRy) \rightarrow (Py \vee \exists z (xRz \wedge Pz)))$$

Let $Pu := \mathbf{u=x}$, we now eliminate $\forall P$:

$$\forall y (xRy \rightarrow \mathbf{y=x} \vee (\exists z (xRz \wedge \mathbf{z=x})))$$

It can be simplified as $\forall y (xRy \rightarrow y = x \vee xRx)$.