Advanced Modal Logic XIV

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Modal and first-order definability over frames

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Definability over frames

If a class of frames can be defined by both a modal formula φ and a first-order sentence α then φ and α are called *frame correspondents*.

There are classes of frames which are modally definable but not first-order definable.

Example: The class of frames defined by Gödel-Löb formula: $\Box(\Box p \to p) \to \Box p$ is *not first-order* definable.

Cf. *Modal Logic for Open Minds* [Ch.21] for some background knowledge about Löb's theorem and provability logic.

Modal and first-order definability

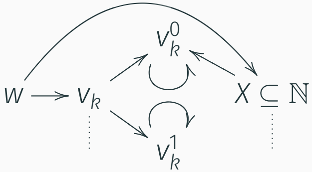
Gödel-Löb formula (GL): $\Box(\Box p \to p) \to \Box p$ defines the class of frames with a binary relation which is transitive and conversely well-founded (no infinite path).

We first show that $\mathcal{F}$ is transitive and conversely well-founded then $\mathcal{F} \vDash GL$. We can also consider its contrapositive: $\Diamond\neg p \to \Diamond(\Box p \wedge \neg p)$.

We then show that $\mathcal{F} \vDash GL \implies \mathcal{F}$ is transitive and conversely well-founded. $V(p) = W/\{w \mid \text{ there is an infinite path from } w\}$.

Modal and first-order definability

However, Löb's formula is not definable by a first-order formula. By an argument based on the compactness of first-order logic (why?). As another example, the class of frames defined by McKinsey formula $\Box\Diamond p \to \Diamond\Box p$ is not first-order definable. (by using the downward Löwenheim-Skolem property of first-order logic, cf. the blue book) The following is a partial model:



Q: if φ does not have any first-order correspondent, is it possible for $\varphi \land \psi$ to have a first-order correspondent? Yes, $\Box\bot$ and Löb's formula together still define the class of isolated irreflexive frames. The conjunction of $\Box p \to \Box\Box p$ and McKinsey formula is a more realistic example.

Frame constructions

We can define the bisimulation notion for frames by dropping the invariance condition on valuation.

- (total) Bisimilarity: does it preserve modal validities?
- Disjoint Union: for all φ: for all $i \in I$ $\mathcal{F}_i \vDash \varphi$ iff $\biguplus_i \mathcal{F}_i \vDash \varphi$.
- Bounded morphism: if $\mathcal{F}'$ is a surjective bounded morphic image of $\mathcal{F}$ then for all φ: $\mathcal{F} \vDash \varphi$ implies $\mathcal{F}' \vDash \varphi$.
- Generated subframe: if $\mathcal{F}'$ is a generated subframe of $\mathcal{F}$ then for all φ: $\mathcal{F} \vDash \varphi$ implies $\mathcal{F}' \vDash \varphi$.
- Ultrafilter extension: for all φ: $\mathfrak{ue}(\mathcal{F}) \vDash \varphi$ implies $\mathcal{F} \vDash \varphi$.
- Q: What about bisimulation contraction and unravelling?

In the proofs of the above preservation results, pay attention to the valuation V and the direction of preservation ("big" to "small")!

Modal and first-order definability

There are classes of frames which are first-order definable but not modally definable.

- Total Connectedness: $\forall xy\,(Rxy \vee Ryx)$, by disjoint union
- Isolation: $\exists x \forall y(\neg Rxy \wedge \neg Ryx)$, by generated subframe
- Irreflexivity: $\forall x \neg Rxx$, also asymmetry, antisymmetry, by surjective bounded morphism
- Successor reflexivity: $\forall x \exists y\, Rxy \wedge Ryy$, non-well-founded, by ultrafilter extension

Suppose a class of frames $\mathtt{K}$ is definable by a FOL-formula, is $\overline{\mathtt{K}}$ also definable by a FOL-formula? Replace FOL by ML in the question, do you have the same answer?

There are also natural frame classes that are neither FO nor modally definable, e.g. acyclicity.

Next:

Which modal formulas have first-order correspondents?
Which first-order definable classes are modally definable?