Advanced Modal Logic XIII

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April 4th, 2023

Advanced Modal Logic (2023 Spring)

Frames

Frames

Frames and validity

Why do we study frames?

- Frames as tools for analysing modal logics (as a set of valid formulas), driven by syntactic approaches to modal logic. Can we characterize a logic by a class of frames via validity (Is system S sound and complete w.r.t. a class of frames)? How can we show $\not\vdash_S \varphi$? From $\forall$ to $\exists$.
- Frames as structures to be described by modal logics via validity. What classes of frames are definable by modal logic? What about its *expressive power over frames* compared with first-order logic?
- Important modal axioms characterize natural frame conditions. The connection gives us a better understanding of both. We can write philosophical theories like mathematical theories.

Models, frames, satisfiability and validity

	local	global	local class	global class
models	$\mathcal{M}, w \vDash \varphi$	$\mathcal{M} \vDash \varphi$	$\mathbb{C}_{pm} \vDash \varphi$	$\mathbb{C}_m \vDash \varphi$
frames	$\mathcal{F}, w \vDash \varphi$	$\mathcal{F} \vDash \varphi$	$\mathbb{C}_{pf} \vDash \varphi$	$\mathbb{C}_f \vDash \varphi$

In terms of the validity over classes of pointed models:

$\mathcal{M}, w \vDash \varphi \iff \{\mathcal{M}, w\} \vDash \varphi$

$\mathcal{M} \vDash \varphi \iff \{\mathcal{M}, w \mid w \in W_{\mathcal{M}}\} \vDash \varphi$

$\mathbb{C}_m \vDash \varphi \iff \{\mathcal{M}, w \mid \mathcal{M} \in \mathbb{C}_m\} \vDash \varphi$

$\mathcal{F}, w \vDash \varphi \iff \{\mathcal{M}, w \mid \mathcal{M} \text{ is based on } \mathcal{F}, w\} \vDash \varphi$

$\mathcal{F} \vDash \varphi \iff \{\mathcal{M}, w \mid \mathcal{M} \text{ is based on } \mathcal{F}\} \vDash \varphi$

$\mathbb{C}_{pf} \vDash \varphi \iff \{\mathcal{M}, w \mid \mathcal{M}, w \text{ is based on } (\mathcal{F}, w) \in \mathbb{C}_{pf}\} \vDash \varphi$

$\mathbb{C}_f \vDash \varphi \iff \{\mathcal{M}, w \mid \mathcal{M} \text{ is based on } \mathcal{F} \in \mathbb{C}_f\} \vDash \varphi$

Models, frames, satisfiability and validity

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φ defines $\mathcal{M}, w$ modulo $\leftrightarroweq$ if for all $\mathcal{N}, v$: $\mathcal{N}, v \vDash \varphi \iff \mathcal{M}, w \leftrightarroweq \mathcal{N}, v$

φ defines $\mathcal{M}$ modulo $\leftrightarroweq$ if for all $\mathcal{N}$: $\mathcal{N} \vDash \varphi \iff \mathcal{M} \leftrightarroweq_{total} \mathcal{N}$

φ defines $\mathbb{C}_{pm}$ if for all $\mathcal{M}, w$: $\mathcal{M}, w \vDash \varphi \iff \mathcal{M}, w \in \mathbb{C}_{pm}$

φ defines $\mathbb{C}_m$ if for all $\mathcal{M}$: $\mathcal{M} \vDash \varphi \iff \mathcal{M} \in \mathbb{C}_m$

φ defines $\mathcal{F}, w$ if for all $\mathcal{F}', w'$: $\mathcal{F}', w' \vDash \varphi \iff \mathcal{F}, w \approx_? \mathcal{F}', w'$

φ defines $\mathcal{F}$ if for all $\mathcal{F}'$: $\mathcal{F}' \vDash \varphi \iff \mathcal{F}' \approx_? \mathcal{F}$

φ defines $\mathbb{C}_{pf}$ if for all $\mathcal{F}, w$: $\mathcal{F}, w \vDash \varphi \iff \mathcal{F}, w \in \mathbb{C}_{pf}$

φ defines $\mathbb{C}_f$ if for all $\mathcal{F}$: $\mathcal{F} \vDash \varphi \iff \mathcal{F} \in \mathbb{C}_f$

Each φ must define a class of models/frames, but may not define any single model or frame.

Models, Frames, satisfiability and validity

φ can be replaced by a set Φ. Natural questions: what kind of …can be define by φ/Φ.

We can also define relative definability (e.g., over frame classes): φ defines $\mathbb{C}$ within $\mathbb{D}$ iff for all $\mathcal{F} \in \mathbb{D}$: $\mathcal{F} \vDash \varphi \iff \mathcal{F} \in \mathbb{C}$.

Q: Is it different from defining the intersection of $\mathbb{D}$ and $\mathbb{C}$?
Q: Is the following right?

$$\begin{aligned} \varphi \text{ defines } \mathbb{C} \quad &\text{iff} \quad \varphi \text{ defines } \{\mathcal{M}, w \mid \mathcal{M} \text{ is based on } \mathcal{F} \in \mathbb{C}\} \\ \mathbb{C} \text{ is modally definable} \quad &\text{iff} \quad \{\mathcal{M}, w \mid \mathcal{M} \text{ is based on } \mathcal{F} \in \mathbb{C}\} \\ &\qquad \text{is modally definable} \end{aligned}$$

Example (modal definability of classes of frames)

$p \to \Diamond p$ defines the class of reflexive frames, i.e., for all $\mathcal{F}$:
$\mathcal{F} \vDash \varphi \iff \mathcal{F}$ is reflexive. ($\Box p \to p$ can also define this class)
$\Box p \to \Diamond p$ defines the class of serial frames (every world has a successor).
$\Box p \to \Box\Box p$ defines the class of transitive frames.
$p \leftrightarrow \Box p$ defines the class of frames which consist of isolated reflexive points.
$\Box\bot$ defines the class of frames which consist of isolated irreflexive points.

Q: Can two non-equivalent modal formulas define the same class of frames? Some tricky things in proving such results: to show $\forall \mathcal{F} : \mathcal{F} \notin \mathbb{C} \implies \mathcal{F} \nvDash \varphi$ we need to find a counter example (model) of φ for each $\mathcal{F} \notin \mathbb{C}$.

Frame Definability

A frame can be viewed as a first-order structure for the language with equality and R_∇ but NO unary predicates P (first-order frame language).

Example (first-order definability of classes of frames)

$\forall x\, xRx$ defines the class of reflexive frames, i.e., for all $\mathcal{F}$:

$\mathcal{F} \Vdash \forall x\, xRx \iff \mathcal{F}$ is reflexive.

$\forall x \exists y\, xRy$ defines the class of serial frames.

$\forall x \forall y\, (Rxy \leftrightarrow x = y)$ defines the class of frames consisting of isolated reflexive points.

$\forall x \forall y\, \neg Rxy$ defines the class of frames consisting of isolated irreflexive points.

Frame and validity

The validity of modal formulas on frames is essentially (monadic) second-order since we need predicate variables over sets of possible worlds. The second-order frame language here is based on the first-order one with a P-indexed collection of monadic predicate variables which can be quantified over.

	local	global
models	$\mathcal{M} \Vdash ST_x(\varphi)[w]$	$\mathcal{M} \Vdash \forall x ST_x(\varphi)$
frames	$\mathcal{F} \Vdash \forall P_1 \cdots \forall P_n ST_x(\varphi)[w]$	$\mathcal{F} \Vdash \forall P_1 \cdots \forall P_n \forall x ST_x(\varphi)$

Frame and validity

Theorem

$\mathcal{F}, w \vDash \varphi \iff \mathcal{F} \Vdash \forall P_1 \cdots \forall P_n ST_x(\varphi)[w]$

$\mathcal{F} \vDash \varphi \iff \mathcal{F} \Vdash \forall P_1 \cdots \forall P_n \forall x ST_x(\varphi)$

Proof.

$(\mathcal{F}, V), w \vDash \varphi \iff \mathcal{F} \Vdash ST_x(\varphi)[w, V(p_1) \cdots V(p_n)]$

□

Digression: Monadic Second Order Logic (MSO)

(Büchi): A language of finite words is recognisable by a finite state automaton if and only if it is MSO definable over finite words. (How do you define the words with even number of symbols?)

(Büchi): A language of infinite words is recognisable by a Nondeterministic Büchi Automaton if and only if it is MSO definable over infinite words.

(Thatcher,Wright) A set of finite trees is recognizable by a finite tree automaton iff it is MSO definable over finite trees.

(Rabin's theorem): MSO over n-successor infinite trees (SnS) is decidable.

(Janin and Walukiewicz): Modal mu-calculus is the bisimulation invariant fragment of MSO.