

Basics in Modal Logic XII

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1 Definability of model class

Characterizing Definability

A class of pointed models K is *definable* by a set of formulas Σ , if for any model \mathcal{M}, w , $\mathcal{M}, w \in K$ iff for any $\phi \in \Sigma$: \mathcal{M}, w satisfies ϕ . A class of pointed models K is definable by a *formula* ϕ if for any model \mathcal{M}, w , $\mathcal{M}, w \in K$ iff \mathcal{M}, w satisfies ϕ . If K is definable by ϕ then \overline{K} is definable by $\neg\phi$. On the other hand, that K is definable by Σ does not imply that \overline{K} is definable by another set of formulas (if Σ is infinite). To see this, suppose K is definable by Σ then $\mathcal{M}, w \in K$ iff for any $\phi \in \Sigma$: \mathcal{M}, w satisfies ϕ . Therefore $\mathcal{M}, w \in \overline{K}$ iff there is a $\phi \in \Sigma$: \mathcal{M}, w satisfies $\neg\phi$. So intuitively the “infinite disjunction” $\bigvee_{\infty} \{\neg\phi \mid \phi \in \Sigma\}$ defines \overline{K} . However, this disjunction may not be expressible by a set of formulas in the logic in concern.

Characterizing definability

A natural question to ask is: when is a class of models definable by a modal formula or a set of modal formulas? A straightforward answer based on van Benthem's theorem:

Proposition

A class of pointed models is definable by a set of modal formulas iff it is definable by a set of bisimulation invariant first-order formulas with one free variable.

More “structural” and informative characterization:

Theorem

A class of pointed models is definable by a set of modal formulas iff it is closed under bisimulation, ultraproduct, and its complement is closed under ultrapower.

Q: Why is there a requirement about the complement of K ?

Characterizing Definability

Proof.

(\Leftarrow) Let $\Sigma = \{\phi \mid \text{for all } \mathcal{M}, w \in K : \mathcal{M}, w \models \phi\}$. We show that Σ defines K . Suppose $\mathcal{M}, w \models \Sigma$, NTS $\mathcal{M}, w \in K$. Suppose not, then $\mathcal{M}, w \in \overline{K}$. Let $\Gamma = \{\psi \mid \mathcal{M}, w \models \psi\}$. Clearly Γ is finitely sat in K (otherwise there is a formula $\neg \bigwedge \{\psi \mid \psi \in \Gamma\} \in \Sigma$). Then Γ is satisfiable in K by some ultraprodut model \mathcal{N}, v (recall the proof of compactness and the fact that K is closed under ultraprodut). Then based on the previous theorems, there exist ultrapowers $\prod_U \mathcal{M}, |f_w| \equiv_{\text{ML}} \prod_{U'} \mathcal{N}, |f_v|$ such that $\prod_U \mathcal{M}$ and $\prod_{U'} \mathcal{N}$ are m -saturated (note that we can always turn a model into m -saturated by some ultrapower). Thus $\prod_U \mathcal{M}, |f_w| \Leftrightarrow \prod_{U'} \mathcal{N}, |f_v|$, however, one is in K and the other in \overline{K} , contradicting to the assumption that K is closed under bisimulation. □

Q: What if we drop the closure condition for \overline{K} ?

Characterizing Definability

Theorem

A class of pointed models is definable by a modal formula iff it and its complement are both closed under bisimulation and ultraproduct.

Proof.

Suppose K and \bar{K} are closed under bisimulation and ultraproduct. From the previous theorem, we have K and \bar{K} defined by Σ_1 and Σ_2 respectively. It is clear that $\Sigma_1 \cup \Sigma_2$ is unsatisfiable. Thus there is a finite subset of $\Sigma_1 \cup \Sigma_2$ is not satisfiable. Therefore there is a formula ϕ_1 (conjunction of some $\phi \in \Sigma_1$) and a formula ϕ_2 (conjunction of some $\psi \in \Sigma_2$) such that $\models \phi_1 \rightarrow \neg\phi_2$. Then K is defined by ϕ_1 . □

A modal characterization

Ultrafilter union is the modal counterpart of ultraproduct. It is essentially an ultrafilter extension of the disjoint union of a family of pointed models $\{\mathcal{M}_i, w_i \mid i \in I\}$ w.r.t. an ultrafilter u such that each cofinite subset of $\{i \in I\}$ belongs to u .

Theorem

A class K of pointed models is definable by a set of modal formula iff K is closed under bisimulation and ultrafilter union and \overline{K} is closed under ultrafilter extension.

Theorem

A class K of pointed models is definable by a modal formula iff K and \overline{K} are closed under bisimulation and ultrafilter union.

Modal model theory: a summary

- We introduced the following notions:
Kripke models, logical equivalence, various structural equivalences, and especially bisimilarity, bisimulation/EF games, saturation, ultrafilter, distinguishing power, expressive power, succinctness, standard translation...
- There are some model construction methods:
bounded morphism, disjoint union, unravelling, generated submodel, bisimulation contraction, filtration, ultrafilter extension, ultraproduct, ultrapower and ultrafilter union.
- We showed that modal logic has:
small model property, decidability, compactness.
- Important result: ν B-Rosen characterization (on finite models), definability characterization. The proof ideas are very important!