Basics in Modal Logic XII

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April. 2nd, 2024

Advanced Modal Logic (2024 Spring)

Definability of model class

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Characterizing Definability

A class of pointed models $\mathbb{K}$ is *definable* by *a set of formulas* Σ, if for any model $\mathcal{M}, w$, $\mathcal{M}, w$ in $\mathbb{K}$ iff for any $\varphi \in \Sigma$: $\mathcal{M}, w$ satisfies φ.
A class of pointed models $\mathbb{K}$ is definable by *a formula* φ if for any model $\mathcal{M}, w$, $\mathcal{M}, w$ in $\mathbb{K}$ iff $\mathcal{M}, w$ satisfies φ.
If $\mathbb{K}$ is definable by φ then $\overline{\mathbb{K}}$ is definable by $\neg\varphi$. On the other hand, that $\mathbb{K}$ is definable by Σ does not imply that $\overline{\mathbb{K}}$ is definable by another set of formulas (if Σ is infinite). To see this, suppose $\mathbb{K}$ is definable by Σ then $\mathcal{M}, w$ in $\mathbb{K}$ iff for any $\varphi \in \Sigma$: $\mathcal{M}, w$ satisfies φ. Therefore $\mathcal{M}, w$ in $\overline{\mathbb{K}}$ iff there is a $\varphi \in \Sigma$: $\mathcal{M}, w$ satisfies $\neg\varphi$. So intuitively the "infinite disjunction" $\bigvee_\infty \{\neg\varphi \mid \varphi \in \Sigma\}$ defines $\overline{\mathbb{K}}$. However, this disjunction may not be expressible by a set of formulas in the logic in concern.

Characterizing definability

A natural question to ask is: when is a class of models definable by a modal formula or a set of modal formulas?A straightforward answer based on van Benthem's theorem:

Proposition

A class of pointed models is definable by a set of modal formulas iff it is definable by a set of bisimulation invariant first-order formulas with one free variable.

More "structural" and informative characterization:

Theorem

A class of pointed models is definable by a set of modal formulas iff it is closed under bisimulation, ultraproduct, and its complement is closed under ultrapower.

Q: Why is there a requirement about the complement of $\mathbb{K}$?

Proof.

($\Leftarrow$) Let $\Sigma = \{\varphi \mid \text{ for all } \mathcal{M}, w \in \mathbb{K} : \mathcal{M}, w \vDash \varphi\}$. We show that Σ defines $\mathbb{K}$. Suppose $\mathcal{M}, w \vDash \Sigma$, NTS $\mathcal{M}, w \in \mathbb{K}$. Suppose not, then $\mathcal{M}, w \in \overline{\mathbb{K}}$. Let $\Gamma = \{\psi \mid \mathcal{M}, w \vDash \psi\}$. Clearly Γ is finitely sat in $\mathbb{K}$ (otherwise there is a formula $\neg \bigwedge\{\psi \mid \psi \in \Gamma\} \in \Sigma$).Then Γ is satisfiable in $\mathbb{K}$ by some ultraproduct model $\mathcal{N}, v$ (recall the proof of compactness and the fact that $\mathbb{K}$ is closed under ultraproduct). Then based on the previous theorems, there exist ultrapowers $\prod_U \mathcal{M}, |f_w| \equiv_{\mathtt{ML}} \prod_{U'} \mathcal{N}, |f_v|$ such that $\prod_U \mathcal{M}$ and $\prod_{U'} \mathcal{N}$ are m-saturated (note that we can always turn a model into m-saturated by some ultrapower). Thus $\prod_U \mathcal{M}, |f_w| \leftrightarroweq \prod_{U'} \mathcal{N}, |f_v|$, however, one is in $\mathbb{K}$ and the other in $\overline{\mathbb{K}}$, contradicting to the assumption that $\mathbb{K}$ is closed under bisimulation. $\square$

Q: What if we drop the closure condition for $\overline{\mathbb{K}}$?

Characterizing Definability

Theorem

A class of pointed models is definable by a modal formula iff it and its complement are both closed under bisimulation and ultraproduct.

Proof.

Suppose $\mathbb{K}$ and $\overline{\mathbb{K}}$ are closed under bisimulation and ultraproduct. From the previous theorem, we have $\mathbb{K}$ and $\overline{\mathbb{K}}$ defined by Σ_1 and Σ_2 respectively. It is clear that $\Sigma_1 \cup \Sigma_2$ is unsatisfiable. Thus there is a finite subset of $\Sigma_1 \cup \Sigma_2$ is not satisfiable.Therefore there is a formula φ_1 (conjunction of some $\varphi \in \Sigma_1$) and a formula φ_2 (conjunction of some $\psi \in \Sigma_2$) such that $\models \varphi_1 \rightarrow \neg\varphi_2$. Then $\mathbb{K}$ is defined by φ_1. $\square$

A modal characterization

Ultrafilter union is the modal counterpart of ultraproduct. It is essentially (a submodel of) the ultrafilter extension of a disjoint union of a family of pointed models $\{\mathcal{M}_i, w_i \mid i \in I\,]$ w.r.t. an appropriate ultrafilter.

Theorem (Venema)

A class $\mathbb{K}$ of pointed models is definable by a set of modal formula iff $\mathbb{K}$ is closed under bisimulation and ultrafilter union and $\overline{\mathbb{K}}$ is closed under ultrafilter extension.

Theorem (Venema)

A class $\mathbb{K}$ of pointed models is definable by a modal formula iff $\mathbb{K}$ and $\overline{\mathbb{K}}$ are closed under bisimulation and ultrafilter union.

Modal model theory: a summary

• We introduced the following notions:
Kripke models, logical equivalence, various structural equivalences, and especially bisimilarity, bisimulation/EF games, saturation, ultrafilter, distinguishing power, expressive power, succinctness, standard translation...

• There are some model construction methods:
bounded morphism, disjoint union, unravelling, generated submodel, bismulation contraction, filtration, ultrafilter extension, ultraproduct, ultrapower and ultrafilter union.

• Modal logic has:
small (tree) model property, decidability, compactness.

• Important result: Hennessy-Milner theorem, vB-Rosen characterization (on finite models), definability characterization. The proof ideas are very important!