



Advanced Modal Logic VII

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Advanced Modal Logic (2023 Spring)

Model Construction Methods

Finite model property

Model Construction Methods

The use of bisimilarity in understanding ML

- to measure the distinguishing power of ML. It preserves the truth values of modal formulas. We can define truth preserving operations on models based on it.
- to understand the *expressive power* of modal logic: which properties of the models can be expressed by modal formulas (modal definability)? A property is *not* definable by the basic modal logic if there are two bisimilar models such that one satisfies the property and one does not.

Example (truth-preserving model construction methods)

- Bounded morphism: $\mathcal{M}, w \Leftrightarrow \mathcal{N}, f(w)$
- Bisimulation contraction: $\mathcal{M}_{\Leftrightarrow}, [w] \Leftrightarrow \mathcal{M}, w$
- Generated submodel: $Gen(\mathcal{M}, X), w \Leftrightarrow \mathcal{M}, w$ ($w \in X$)
- Disjoint Union: $\biguplus_i \mathcal{M}_i, w_i \Leftrightarrow \mathcal{M}_i, w_i$
- Unravelling: $Unr(\mathcal{M}, w), \langle w \rangle \Leftrightarrow \mathcal{M}, w$

Bisimulation contraction

Definition (Bisimulation contraction)

Given a Kripke model \mathcal{M} of a unary similarity type, the bisimulation contraction of \mathcal{M} is the *quotient model*

$\mathcal{M}_{\Leftrightarrow} = \langle W', \{\rightarrow^a\}_{a \in \mathcal{O}}, V' \rangle$ where:

- $W' = \{[w] \mid w \in W_{\mathcal{M}}\}$ where $[w] = \{v \mid w \Leftrightarrow v\}$
- $[w] \xrightarrow{a} [v]$ iff $\exists w' \in [w] \exists v' \in [v]$ such that $w' \xrightarrow{a} v'$ in \mathcal{M}
- $V'([w]) = V(w)$ (the choice of w *does not matter*)

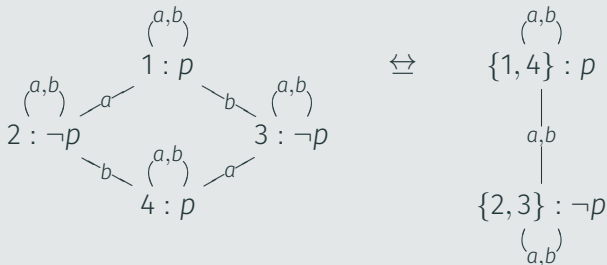
It is also fine to replace $\exists \exists$ by $\forall \exists$ in the definition of \xrightarrow{a} .

By defining a bisimulation, we can show that

$$\mathcal{M}_{\Leftrightarrow}, [w] \Leftrightarrow \mathcal{M}, w$$

Bisimulation contraction

Example (Making models smaller!)



Therefore counting modalities is not *definable* on arbitrary models: e.g., $\mathcal{M}, w \models E_4\varphi \iff$ there are exactly 4 worlds in \mathcal{M} that satisfy φ . Q: Is bisimulation contraction the smallest model that is total-bisimilar to the original model?

Generated submodel

Definition (Submodel)

Given a Kripke model \mathcal{M} of a unary similarity type, a submodel \mathcal{M}' of \mathcal{M} is a Kripke model such that $W_{\mathcal{M}'} \subseteq W_{\mathcal{M}}$ and for all $w', w'' \in W_{\mathcal{M}'}$: $w' \xrightarrow{a}_{\mathcal{M}'} w'' \iff w' \xrightarrow{a}_{\mathcal{M}} w''$ (i.e., $\xrightarrow{a}_{\mathcal{M}'} = \xrightarrow{a}_{\mathcal{M}}|_{W_{\mathcal{M}'} \times W_{\mathcal{M}'}}$), and for all $w \in W_{\mathcal{M}'}$: $V_{\mathcal{M}'}(w) = V_{\mathcal{M}}(w)$

Definition (Generated submodel)

Given a Kripke model \mathcal{M} of a unary similarity type, a generated submodel \mathcal{M}' of \mathcal{M} is a submodel of \mathcal{M} such that if $w \xrightarrow{a}_{\mathcal{M}} w'$ and $w \in W_{\mathcal{M}'}$ then $w' \in W_{\mathcal{M}'}$ (closed under transitions).

If \mathcal{M}' is a generated submodel of \mathcal{M} then for all $w \in W_{\mathcal{M}'}$:
 $\mathcal{M}', w \Leftrightarrow \mathcal{M}, w$.

Definition (Generated submodel w.r.t. X)

Given a Kripke model \mathcal{M} of a unary similarity type, the generated submodel \mathcal{M}' of \mathcal{M} w.r.t. $X \subseteq W_{\mathcal{M}}$ is the smallest submodel of \mathcal{M} such that $X \subseteq W_{\mathcal{M}'}$ and if $(w \xrightarrow{a}_{\mathcal{M}} w'$ and $w \in W_{\mathcal{M}'})$ then $w' \in W_{\mathcal{M}'}$. If X is a singleton then the generated submodel is said to be point generated.

Is backward looking modality definable? $\mathcal{M}, w \vDash \langle a \rangle^{\leftarrow} \varphi \iff$
there exists v such that $v \xrightarrow{a} w$ and $\mathcal{M}, v \vDash \varphi$.

Disjoint Union

Two models are *disjoint* if they do not share a common possible world.

Definition (Disjoint Union)

Given a set of *mutually disjoint* Kripke models $\{\mathcal{M}_i \mid i \in I\}$ of a unary similarity type, the *disjoint union* of \mathcal{M}_i is a Kripke model $\biguplus_{i \in I} \mathcal{M}_i$ such that

- $W_{\biguplus_{i \in I} \mathcal{M}_i} = \bigcup_{i \in I} W_{\mathcal{M}_i}$
- for all $\Box_a \in \mathbf{O}$: $\xrightarrow{a}_{\biguplus_{i \in I} \mathcal{M}_i} = \bigcup_{i \in I} \xrightarrow{a}_{\mathcal{M}_i}$
- $V_{\biguplus_{i \in I} \mathcal{M}_i}(w) = V_{\mathcal{M}_i}(w)$ if $w \in W_{\mathcal{M}_i}$

What if \mathcal{M}_i are not disjoint? We then take isomorphic copies. For any $w \in W_{\mathcal{M}_j}$: $\biguplus_{i \in I} \mathcal{M}_i, w \leftrightarrow \mathcal{M}_j, w$. Definability?

(Tree) unravelling

Definition (Unravelling)

Given a pointed model \mathcal{M}, w of a unary similarity type, the *unravelling* of \mathcal{M} at w is a pointed model $Unr(\mathcal{M}, w), \langle w \rangle$ s.t.

- $W_{Unr(\mathcal{M}, w)} = \{s \mid s = w_0 w_1 \dots w_k \text{ such that } w_0 = w \text{ and for } 0 \leq i < k: w_i \xrightarrow{a} w_{i+1} \text{ for some } \square_a \in \mathbf{O} \}$
- for all $\square_a \in \mathbf{O}$: $s \xrightarrow{a}_{Unr(\mathcal{M}, w)} s'$ if $s = w \dots w_k$ and $s' = s w_{k+1}$ and $w_k \xrightarrow{a}_{\mathcal{M}} w_{k+1}$
- $V_{Unr(\mathcal{M}, w)}(w_0 \dots w_k) = V_{\mathcal{M}}(w_k)$

In the above definition, we allow multiple transitions from one world to another, e.g., $s \xrightarrow{a} t$ and $s \xrightarrow{b} t$. It is fine for most applications of unravelling. To rule out such cases, we can define the set of worlds as the mixed sequences of worlds and labels, e.g., $w_0 a_0 w_1 a_1 \dots w_k$. $Unr(\mathcal{M}, w), \langle w \rangle \Leftrightarrow \mathcal{M}, w$.

Finite model property

Definition (Finite model property)

A logic $\langle L, \mathbb{C}, \models \rangle$ has the *finite model property* if for any φ in this logical language: φ is satisfiable at a model in \mathbb{C} implies it is satisfiable at a finite model ($W_{\mathcal{M}}$ is finite) in \mathbb{C} .

First-order logic on the class of arbitrary models does not have finite model property:

e.g., $\forall x \neg(x < x) \wedge \forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z) \wedge \forall x \exists y (x < y)$
has only infinite models.

A set of modal formulas may only have infinite models, e.g.,
 $\{\diamond \Box \perp, \diamond \Box \Box \perp, \diamond \Box \Box \Box \perp \dots\}$ where $\Box^n \varphi := \Box \varphi \wedge \Box^n \varphi$.

Finite model property

Finite model property over arbitrary models

Theorem (ML has the finite model property)

If a modal logic formula is satisfiable then it is satisfiable in a finite model.

Proof.

Given a formula φ , suppose it is satisfiable at a pointed model \mathcal{M}, w . Let $k = \text{deg}(\varphi)$, we cut $\text{Unr}(\mathcal{M}, w)$ up to “level” k (call the resulting model $\text{Unr}(\mathcal{M}, w)|_k$). We can show $\mathcal{M}, w \Leftrightarrow_k \text{Unr}(\mathcal{M}, w)|_k$. Then we prune the branches according to ML_k equivalence (over finitely many basic propositions in φ) such that it is finitely branching. \square

Therefore ML **cannot** define classes consisting of only infinite models.