Advanced Modal Logic VI

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Advanced Modal Logic (2024 Spring)

Bisimilarity and modal equivalence

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Correspondence between $\underline{\leftrightarrow}_n$ and $\equiv_{\mathtt{ML}_n}$

Modal degree (modality depth):

$$
\begin{array}{rclrcl}
deg(\top) & = & 0 & deg(p) & = & 0 \\
deg(\neg\varphi) & = & deg(\varphi) & deg(\varphi \wedge \psi) & = & MAX((deg(\varphi), deg(\psi)) \\
deg(\Box\varphi) & = & deg(\varphi) + 1 & & &
\end{array}
$$

Let $\mathtt{ML}_n$ be the fragment of $\mathtt{ML}$ consisting of all the formulas of modal degree *at most n*.

Correspondence between $\leftrightarroweq_n$ and $\equiv_{\mathtt{ML}_n}$

Proposition

Modulo modal logical equivalence, there are only finitely many different formulas in $\mathtt{ML}_n$ (assuming a finite P *and a finite unary similarity type τ).*

A is the same as B *modulo* C means, more-or-less, A and B are the same except for differences not explained by C.

φ is modally equivalent to ψ if $\vDash \varphi \leftrightarrow \psi$.

Proof.

Note that $\mathtt{ML}_0$ is

$$\varphi ::= p \mid (\varphi \wedge \varphi) \mid \neg\varphi$$

$\mathtt{ML}_{k+1}$ is:

$$\varphi ::= \psi \mid \Box_a \psi \mid (\varphi \wedge \varphi) \mid \neg\varphi$$

where $\psi \in \mathtt{ML}_k$. Prove inductively they are essentially finite. □

Approximations of bisimilarity

Definition (n-bisimilarity $\leftrightarroweq_n$)

1. for any $\mathcal{M}, w, \mathcal{N}, v$: $\mathcal{M}, w \leftrightarroweq_0 \mathcal{N}, v$ iff $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$
2. $\mathcal{M}, w \leftrightarroweq_{n+1} \mathcal{N}, v$ if:
    - $\mathcal{M}, w \leftrightarroweq_n \mathcal{N}, v$
    - for any a, if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there is a v' such that $v \xrightarrow{a} v'$ and $\mathcal{M}, w' \leftrightarroweq_n \mathcal{N}, v'$
    - for any a, if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there is a w' such that $w \xrightarrow{a} w'$ and $\mathcal{M}, w' \leftrightarroweq_n \mathcal{N}, v'$

Correspondence between $\leftrightarroweq_n$ and $\equiv_{\mathtt{ML}_n}$

Theorem (Given finite P and τ)

For any $\mathcal{M}, w, \mathcal{N}, v$: $\mathcal{M}, w \leftrightarroweq_n \mathcal{N}, v \iff \mathcal{M}, w \equiv_{ML_n} \mathcal{N}, v$.

$\Rightarrow$: induction on n and case $n = 0$ is trivial.

- IH: For $n = k$ for any $\mathcal{M}, w, \mathcal{N}, v$: $\mathcal{M}, w \leftrightarroweq_n \mathcal{N}, v \implies \mathcal{M}, w \equiv_{\mathtt{ML}_n} \mathcal{N}, v$
- $n = k + 1$: Suppose $\mathcal{M}, w \leftrightarroweq_{k+1} \mathcal{N}, v$ (therefore $\mathcal{M}, w \leftrightarroweq_k \mathcal{N}, v$). Now we prove by another induction on $\varphi \in \mathtt{ML}_{k+1}$ that $\mathcal{M}, w \equiv_{\mathtt{ML}_{k+1}} \mathcal{N}, v$. The only crucial part is when $\varphi = \Box_a \psi$ where ψ is an $\mathtt{ML}_k$ formula.

$\Box$

Note that the proof of this direction does not use the finitary condition.

Correspondence between $\underline{\leftrightarrow}_n$ and $\equiv_{\texttt{ML}_n}$

Theorem (Given finite P and τ)

For any $\mathcal{M}, w$, $\mathcal{N}, v$: $\mathcal{M}, w \underline{\leftrightarrow}_n \mathcal{N}, v \iff \mathcal{M}, w \equiv_{ML_n} \mathcal{N}, v$.

$\Leftarrow$: Induction on n and case $n = 0$ is trivial.

- IH: For $n = k$: $\mathcal{M}, w \equiv_{\texttt{ML}_n} \mathcal{N}, v \implies \mathcal{M}, w \underline{\leftrightarrow}_n \mathcal{N}, v$.
- $n = k + 1$: Suppose $\mathcal{M}, w \equiv_{\texttt{ML}_{k+1}} \mathcal{N}, v$ (therefore $\mathcal{M}, w \equiv_{\texttt{ML}_k} \mathcal{N}, v$). By IH $\mathcal{M}, w \underline{\leftrightarrow}_k \mathcal{N}, v$. We only need to check the zig-zag conditions. Suppose $w \xrightarrow{a} w'$ then there is a formula $\Diamond\varphi$ such that φ is equivalent to $\bigwedge\{\psi \mid \mathcal{M}, w' \vDash \psi \text{ and } \psi \in \texttt{ML}_k\}$ (we only need finitely many "representative" ψ modulo equivalence) and $\mathcal{M}, w \vDash \Diamond\varphi$. Since $\mathcal{M}, w \equiv_{\texttt{ML}_{k+1}} \mathcal{N}, v$ then $\mathcal{N}, v \vDash \Diamond\varphi$ therefore there is a v' such that $v \xrightarrow{a} v'$ and $\mathcal{M}, w' \equiv_{\texttt{ML}_k} \mathcal{N}, v'$. By induction hypothesis $\mathcal{M}, w' \underline{\leftrightarrow}_k \mathcal{N}, v'$. Therefore $\mathcal{M}, w \underline{\leftrightarrow}_{k+1} \mathcal{N}, v$.

$\square$

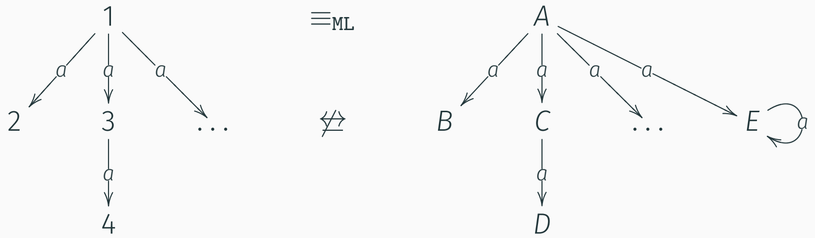
Correspondence between $\underline{\leftrightarrow}_\omega$ and $\equiv_{\mathtt{ML}}$

Corollary (Given finite P and τ)

$\mathcal{M}, w \underline{\leftrightarrow}_\omega \mathcal{N}, v \iff \mathcal{M}, w \equiv_{ML} \mathcal{N}, v$

Proposition (No finite resitriction)

$\mathcal{M}, w \underline{\leftrightarrow} \mathcal{N}, v \implies \mathcal{M}, w \equiv_{ML} \mathcal{N}, v$



Correspondence between $\leftrightarroweq$ and $\equiv_{\mathtt{ML}}$

Theorem (Hennessy-Milner Theorem, no finite restriction on P)

On image-finite models $\mathcal{M}, w \leftrightarroweq \mathcal{N}, v \iff \mathcal{M}, w \equiv_{ML} \mathcal{N}, v$

$\Leftarrow$: not an inductive proof!

Let $Z \subseteq W_{\mathcal{M}} \times W_{\mathcal{N}}$ be defined as $\{(w, v) \mid w \equiv_{\mathtt{ML}} v\}$. We need to show that Z is a bisimulation. The propositional invariance condition holds trivially. Now suppose $w \xrightarrow{a} w'$ we need to show there is a v' such that $v \xrightarrow{a} v'$ in $\mathcal{N}$ and $v \equiv_{\mathtt{ML}} v'$. Suppose not, then for each a-successor v' of v we have $\varphi_{w',v'}$ holds at $\mathcal{M}, w'$ but it is false at $\mathcal{N}, v'$. Take the finite conjunction of such $\varphi_{w',v'}$ (due to the image-finiteness) and call it ψ.Then $\Diamond\psi$ holds at $\mathcal{M}, w$ but not at $\mathcal{N}, v$, contradiction. (Also consider the case when there is no successor of v). $\square$

Correspondence between $\leftrightarroweq$ and $\equiv_{\mathtt{ML}}$

Theorem (Hennessy-Milner Theorem)

On image-finite models $\mathcal{M}, w \leftrightarroweq \mathcal{N}, v \iff \mathcal{M}, w \equiv_{ML} \mathcal{N}, v$

A class of models has the *Hennessy-Milner property* if $\mathcal{M}, w \leftrightarroweq \mathcal{N}, v \iff \mathcal{M}, w \equiv_{\mathtt{ML}} \mathcal{N}, v$ for any $\mathcal{M}, \mathcal{N}$ in this class and any w in $\mathcal{M}$ and v in $\mathcal{N}$. The classes of models which have this property are also called *Hennessy-Milner classes*.

Q: Is the class of image-finite models the "largest" H-M class?

Correspondence between $\leftrightarroweq$ and $\equiv_{\mathtt{ML}}$

A set of formulas Σ is *finitely satisfiable* in a set of worlds X in a model if *any* finite subset of Σ is satisfiable in X.

Definition (m-saturation)

A Kripke model of a unary similarity type is m-saturated if at *each* world w, for *each* $a \in \tau$, if a set of formulas is finitely satisfiable in the set of a-successors of w then it is satisfiable at some a-successor of w.

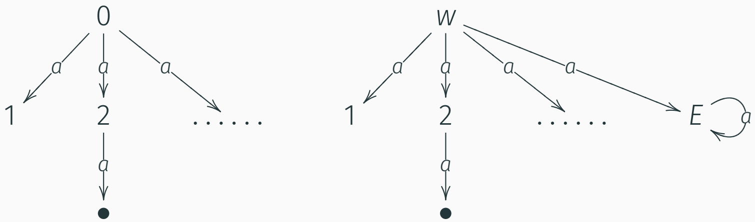
Proposition

The class of image-finite models is a subclass of the class of m-saturated models.

Note that in m-saturated models, if a set of formulas is not satisfiable at the a-successors of some world then there is a finite set of it which is not satisfiable at those a-successors.

Correspondence between $\leftrightarroweq$ and $\equiv_{\mathtt{ML}}$

The set of formulas $\Sigma = \{\langle a\rangle\top, \langle a\rangle\langle a\rangle\top, \langle a\rangle\langle a\rangle\langle a\rangle\top, \dots\}$ is finitely satisfiable in the set of successors of 0, but Σ itself is not satisfiable at any successor of 0. On the other hand, Σ is satisfiable at E in the model on the right.



Is the right-hand model really m-saturated? (exercise)

Correspondence between $\leftrightarroweq$ and $\equiv_{\texttt{ML}}$

Theorem (m-saturation)

The class of m-saturated models is a Hennessy-Milner class.

(From logical equivalence to bisimulation).

Let $R \subseteq W_{\mathcal{M}} \times W_{\mathcal{N}}$ be defined as $\{(w, v) \mid w \equiv_{ML} v\}$. We need to show that R is a bisimulation. Now suppose $w \xrightarrow{a} w'$ in $\mathcal{M}$ we need to show there is a v' such that $v \xrightarrow{a} v'$ in $\mathcal{N}$ and $w' \equiv_{\texttt{ML}} v'$. Suppose not, then for each a successor v' of v we have $\varphi_{w',v'}$ holds at $\mathcal{M}, w'$ but it is false at $\mathcal{N}, v'$. Let Φ be the set of such formulas. Clearly Φ is not satisfiable at any successor of v. Since $\mathcal{N}$ is m-saturated, then there is a finite set Φ' of Φ that is not satisfiable at any successor of v. Then $\Diamond \bigwedge \Phi'$ holds at $\mathcal{M}, w$ but not at $\mathcal{N}, v$, contradiction. $\square$

Summary

Structural equivalence	Game	Logic
$\underline{\leftrightarrow}_n$	G_n	$\mathtt{ML}_n$ (finite P, τ)
$\underline{\leftrightarrow}_\omega$ ($\bigcap_n \underline{\leftrightarrow}_n$)	all the G_n	$\mathtt{ML}$ (finite P, τ)
$\underline{\leftrightarrow}$ (on m-saturated models)	G_∞	$\mathtt{ML}$
$\underline{\leftrightarrow}$	G_∞	$\mathtt{ML}_\infty$

You will generalize the results to polyadic modal logic.

Where $\mathtt{ML}_\infty$ is defined as follows:

$$\varphi \quad ::= \quad \top \mid p \mid \neg\varphi \mid \bigwedge \Phi \mid \Box\varphi$$

where Φ is a set of $\mathtt{ML}_\infty$ formulas. Modal degree by ordinals:

$$\begin{aligned} deg(\top) &= 0 & deg(p) &= 0 \\ deg(\neg\varphi) &= deg(\varphi) & deg(\bigwedge \Phi) &= sup(\{deg(\varphi) \mid \varphi \in \Phi\}) \\ deg(\Box\varphi) &= deg(\varphi) + 1 \end{aligned}$$

$\mathtt{ML}_\infty \implies \underline{\leftrightarrow}$ can be proved by deriving a contradiction assuming no corresponding successor as in the case of m-saturation.