Advanced Modal Logic V

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Advanced Modal Logic (2024 Spring)

The discovery of bisimulation

- In modal logic: van Benthem (1976) (based on the work of Segerberg (1971), de Jongh and Troelstra (1966) + the insight of a relational definition)
- In Computer Science: Park (1981) (based on the work of Milner (1980) + the insight of the greatest fixed point)
- In (non-well-founded) Set theory: Forti and Honsell (1981) Hinnion, and it was made popular by Aczel (1988)

Non-wellfounded set theory

Foundation Axiom (FA) implies that there is no infinite "descending" sequence of sets:

$$\cdots \in x_3 \in x_2 \in x_1 \in x_0$$

Two sets are equivalent iff they contain the same elements.

$$A = \{B\}, B = \{A\}, C = \{C, A\}$$

Are A, B and C equivalent?

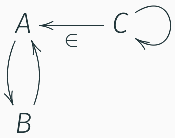
Basic intuition: if x and y are considered equivalent then *at least* the following should hold:

- for each element x' of x there is an element y' of y such that x' and y' are also considered equivalent,
- for each element y' of y there is an element y' of y such that x' and y' are also considered equivalent.

Non-wellfounded set theory to modal logic

$$A = \{B\}, B = \{A\}, C = \{C, A\}$$

The sets and the membership relation can be viewed as a graph:



The total relation $Z = \{A, B, C\} \times \{A, B, C\}$ is indeed a bisimulation relation, thus A, B, C are bisimilar to each other. The sets $A = \{B\}, B = \{A\}, C = \{C, A\}$ can be viewed equivalent.

Further readings

- D. Sangiorgi, On the origins of Bisimulation and Coinduction. *ACM Transactions on Programming Languages and Systems*, Vol. 31, No. 4, 2009.
- D. Sangiorgi, Introduction to Bisimulation and Coinduction. Cambridge University Press, 2011

Circularity in dictionaries

Merriam-Webster dictionary (2013):

- Hill (1): “a usually rounded natural elevation of land **lower than a mountain**”
- Mountain (1a): “a landmass that projects conspicuously above its surroundings and is **higher than a hill**”

- Oak (1a): “any of a genus (Quercus) of trees or shrubs of the beech family that produce **acorns**”
- Acorn: “the nut of the **oak tree**”

We can still understand those words due to their connections with other words. Similar situations are ubiquitous, e.g., importance of webpage (Google’s pagerank).

We can use *n*-bisimulation to measure the ‘distance’ between words, or capture formal analogy.

The discovery of bisimulation

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Yet another perspective to look at bisimulaiton is through *games*. Games and Logic have crucial connections: logic games (and game logic). Game trees can be viewed as Kripke structures and there are logic to reason about them, e.g., Alternating-time temporal logic (ATL). Games can also be used to capture interactive computations...

Bisimulation Games

Bisimulation Games

Abstract games

A (typical turn-based) *game* consists of:

- a set of players: I
- a set of configurations (with an initial one): C
- a player assignment $f : C \to I$
- (deterministic) actions connecting configurations
- number of runs
- winning/losing conditions

A (memory-less) *strategy* of a player for a game is a function assigning to each configuration that belongs to him or her a move to proceed (if possible). We assume all the information about the game before and during the plays are known to the players (*perfect information* games), in particular the current configuration is known to the players during a play.

Determinacy of a game

A game is said to be *determined* if one of the players has a *winning strategy* (guarantees winning, no matter what others may do).

Theorem (Zemelo? Von Neumann?)

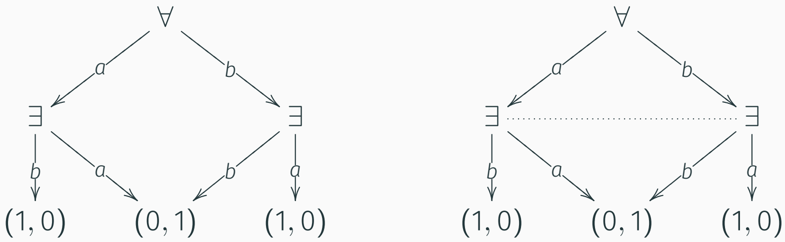
2-player finite-depth deterministic perfect-information win/lose games are determined.

Proof 1: bottom-up labelling the winning configurations.

Proof 2: the first player has a winning strategy iff $\exists a_0 \forall b_0 \exists a_1 \forall b_1 \ldots \exists a_n \forall b_n : \langle a_0 b_0 \cdots a_n b_n \rangle \in WC(1)$ then the first player does not have a winning strategy iff $\neg(\exists a_0 \forall b_0 \exists a_1 \forall b_1 \ldots \exists a_n \forall b_n : \langle a_0 b_0 \cdots a_n b_n \rangle \in WC(1))$iff $\forall a_0 \exists b_0 \forall a_1 \ldots \forall a_n \exists b_n : \langle a_0 b_0 \cdots a_n b_n \rangle \notin WC(1)$ iff Player 2 has a winning strategy. Excluded middle implies determinacy. $\square$

Determinacy of a game

What about non-deterministic actions? What about three players? What about imperfect information games?



$$\forall x \exists y\ x = y \text{ v.s. } \forall x \exists y_{/\forall x} x = y$$

∀ and ∃ are often called Abelard and Eloise (Heloise) in game semantics (from Hodges).

Design a game semantics for modal logic (exercise).

Chess

Chess is a finite game (given the regulation on draw). Note that White always moves first. Let White-Chess be the game just like normal Chess but count a draw in the normal game as a win for white, similar for Black-Chess.

White-Chess	Black-Chess	Chess
White has a w.s.	White has a w.s.	White has a w.s
Black has a w.s.	White has a w.s.	Impossible
White has a w.s.	Black has a w.s.	Both have non-losing strategies
Black has a w.s.	Black has a w.s.	Black has a w.s.

Corollary (Zemelo 1913)

In Chess, either White can force a win, or Black can force a win, or both sides can force a draw

Bisimulation game

Definition (n-round Bisimulation Game)

An n-round bisimulation game $\mathcal{G}_n((\mathcal{M}, w), (\mathcal{N}, v))$ between $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ is a two player game based on the configurations in $W_{\mathcal{M}} \times W_{\mathcal{N}}$. The initial configuration is (w, v) and the players, Spoiler and Defender, play in rounds. At each configuration (w', v'):

1. Spoiler selects a state w'' in $\mathcal{M}$ such that $w' \xrightarrow{a}_{\mathcal{M}} w''$ for some $a \in \mathsf{O}$ and then Defender needs to select a state v'' in $\mathcal{N}$ such that $v' \xrightarrow{a}_{\mathcal{N}} v''$. The configuration is then changed to (w'', v'').
2. Similar for the case when Spoiler first selects a state v'' in $\mathcal{N}$.

To be extremely precise we need to define configurations as $W \times W \times \{S, D\} \times \mathsf{O} \times \{1, ..., n\}$ and then the strategies will be indeed functions on configurations.

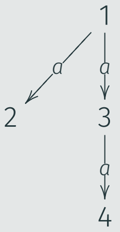
Bisimulation game

Spoiler wins the game if within n rounds some configuration (w', v') is reached such that $V_{\mathcal{M}}(w') \neq V_{\mathcal{N}}(v')$ or it is Defender's turn but she does not have a legal move to do. Defender wins the game otherwise, i.e., either Spoiler gets stuck at some point, or because Defender can respond with legal moves for the duration of the game.

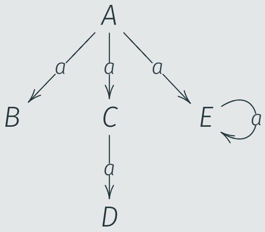
We say that Defender has a *winning strategy* in the n-round bisimulation game, if she has responses to any challenges from Spoiler that guarantee her to win the game. Winning strategies of Spoiler is defined similarly.

Bisimulation Game

Example



$\underline{\leftrightarrow}_2$



Defender can win k-round games ($k < 3$)

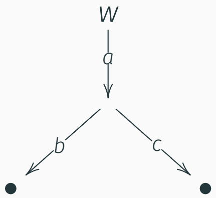
$\underline{\leftrightarrow}_0 = \{1, 2, 3, 4\} \times \{A, B, C, D, E\}$

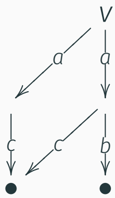
$\underline{\leftrightarrow}_1 = \{(2, B), (2, D), (4, B), (4, D), (1, A), (1, C), (1, E), (3, A), (3, C), (3, E)\}$

$\underline{\leftrightarrow}_2 = \{(2, B), (2, D), (4, B), (4, D), (1, A), (3, C)\}$

$\underline{\leftrightarrow}_3 = \{(2, B), (2, D), (4, B), (4, D), (3, C)\}$ so $1 \not\underline{\leftrightarrow}_3 A$

Bisimulation Game





Spoiler can win the two round bisimulation game. It is important to switch models!

Approximations of bisimilarity

Definition (n-bisimilarity $\leftrightarroweq_n$)

1. for any $\mathcal{M}, w, \mathcal{N}, v$: $\mathcal{M}, w \leftrightarroweq_0 \mathcal{N}, v$ iff $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$
2. $\mathcal{M}, w \leftrightarroweq_{n+1} \mathcal{N}, v$ if:
   - $\mathcal{M}, w \leftrightarroweq_n \mathcal{N}, v$
   - for any a, if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there is a v' such that $v \xrightarrow{a} v'$ and $\mathcal{M}, w' \leftrightarroweq_n \mathcal{N}, v'$
   - for any a, if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there is a w' such that $w \xrightarrow{a} w'$ and $\mathcal{M}, w' \leftrightarroweq_n \mathcal{N}, v'$

Approximations of bisimilarity

Definition (n-bisimilarity $\underline{\leftrightarrow}_n$: an equivalent definition)

1. for any $\mathcal{M}, w, \mathcal{N}, v$: $\mathcal{M}, w \underline{\leftrightarrow}_0 \mathcal{N}, v$ iff $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$
2. $\mathcal{M}, w \underline{\leftrightarrow}_{n+1} \mathcal{N}, v$ if:
    - $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$
    - for any a, if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there is a v' such that $v \xrightarrow{a} v'$ and $\mathcal{M}, w' \underline{\leftrightarrow}_n \mathcal{N}, v'$
    - for any a, if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there is a w' such that $w \xrightarrow{a} w'$ and $\mathcal{M}, w' \underline{\leftrightarrow}_n \mathcal{N}, v'$

Bisimulation game

Theorem (Adequacy)

For all $n \in \mathbb{N}$, $\mathcal{M}, w \leftrightarroweq_n \mathcal{N}, v$ iff Defender has a winning strategy of the n-round bisimulation game $\mathcal{G}_n((\mathcal{M}, w), (\mathcal{N}, v))$.

Proof: by induction on n

- $n = 0$: Defender has a winning strategy of the 0-round bisimulation game iff $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$ iff $\mathcal{M}, w \leftrightarroweq_0 \mathcal{N}, v$
- Inductive hypothesis (IH): for $n \leq k$ the statement holds.

to be continued...

Bisimulation game

cont.

- $n = k + 1$: $\Leftarrow$: Suppose Defender has a winning strategy in the $k+1$-round game then she has a winning strategy in the k-round game. From IH we have $\mathcal{M}, w \underline{\leftrightarrow}_k \mathcal{N}, v$. Now we check the "Zig-Zag" conditions. Suppose $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there is a v' such that $v \xrightarrow{a} v'$ and (w', v') is also a winning configuration for Defender in the next k rounds (since Defender has a winning strategy in the $k+1$-round game). By IH $w' \underline{\leftrightarrow}_k v'$. Similar for the case of $v \xrightarrow{a} v'$ in $\mathcal{N}$. Therefore $\mathcal{M}, w \underline{\leftrightarrow}_{k+1} \mathcal{N}, v$.

  $\Rightarrow$: a winning strategy for Defender starts with: " try to keep the $\underline{\leftrightarrow}_k$ pairs of worlds then keep the $\underline{\leftrightarrow}_{k-1}$ pairs ..." (Axiom of choice is needed here: note that although the number of rounds is bounded, the possible successors to choose may be infinite for both Spoiler and Defender.)

Bisimulation game

Theorem

$\mathcal{M}, w \leftrightarroweq_\omega \mathcal{N}, v$ iff Defender has a winning strategy in the n-round bisimulation game $\mathcal{G}_n((\mathcal{M}, w), (\mathcal{N}, v))$ for each n.

What about playing the game infinitely long?

The winning condition for Defender in the infinite game is simply: "playing forever (if Spoiler had not already lost)".

Infinite Bisimulation Game

Theorem

$\mathcal{M}, w \leftrightarroweq \mathcal{N}, v$ iff Defender has a winning strategy in the infinite bisimulation game $\mathcal{G}_\infty((\mathcal{M}, w), (\mathcal{N}, v))$.

Proof.

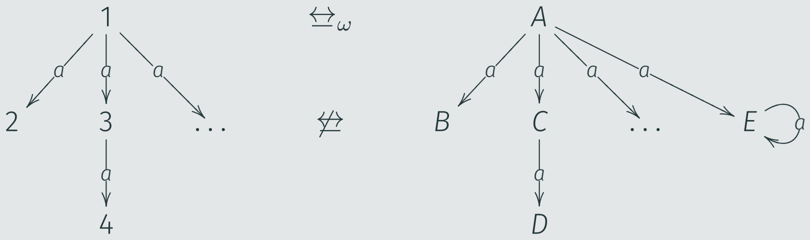
$\Rightarrow$: The winning strategy is: "Keep the bisimilar pairs". (Here you need the axiom of choice…)
$\Leftarrow$: let $Z = \{(w', v') \mid$ it is Spoiler's turn to move at (w', v') and (w', v') is a winning position for Defender $\}$. Here a winning position means that Defender can win if the game continues from this configuration. Show that Z is a bisimulation. $\square$

Corollary

$\mathcal{M}, w \leftrightarroweq \mathcal{N}, v$ implies $\mathcal{M}, w \leftrightarroweq_\omega \mathcal{N}, v$.

Bisimulation and ω-bisimulation

Example



Can you find other examples (about $\leftrightarroweq_{\omega^2}$ and $\leftrightarroweq$)?

BTW Bisimulation games are also determined.

Games in Logic

A statement is true iff a particular player (usually “defender ”) has a winning strategy in a game.

- Game semantics (model checking games) ($\mathcal{M}, w \vDash \varphi$)
- Model comparison games (e.g., $\mathcal{M}, w \leftrightarroweq \mathcal{N}, v$). In FOL, Ehrenfeucht-Fraïssé games
- Satisfiability games ($\vDash \neg\varphi$)

Connections of logic, games, tableau, and automata.

Summary

Bisimulation	Game
$\underline{\leftrightarrow}_n$	G_n
$\underline{\leftrightarrow}_\omega$ ($\bigcap_n \underline{\leftrightarrow}_n$)	all the G_n
$\underline{\leftrightarrow}$	G_∞

What about the logical equivalence?

Dialogue games

Lorenzen Game (1958)

Given a formula φ in the language of propositional logic with $\neg, \vee, \to$ and $\wedge$.

- Players: **P**roponent and **O**pponent.
- Configurations: φ and its subformulas.
- Rules:
  - P first *asserts* φ, then P and O play in turns:
  - each round O can Attack or Defend the last round.
  - each round P can Attack or Defend any previous rounds.
  - To assertion $\psi \vee \psi'$, "which one?" is an attack.
  - To assertion $\psi \wedge \psi'$ "left?" "right?" are attacks
  - To assertion $\psi \to \psi'$ or $\neg\psi$, "what if? assert(ψ)" is an attack
  - Facing attacks, one can use assertion to defend (except $\neg\varphi$). O can assert any atomic proposition, P can assert the atomic propositions that O asserted previously, (non-atomic can be asserted by anyone).

Example (from Benedikt Löwe)

If one cannot move then the other wins.

0		–		assert($\neg\neg p \rightarrow p$)
1	attack(0)	what if? assert($\neg\neg p$)		
2			attack(1)	what if? assert($\neg p$)
3	attack(2)	what if? assert(p)		
4			defend(1)	assert(p)
5	–	–		

You can try other formulas:

- $p \rightarrow p$
- $p \vee \neg p$
- $p \rightarrow (q \rightarrow p)$
- $(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow q)$