Advanced Modal Logic IV

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Advanced Modal Logic (2023 Spring)

Bisimulation

Bisimulation

Recap: the notion of sameness

The discovery of bisimulation in computer science:

- When two processes can be viewed the same?
- Inductively defined structural equivalences that can go finer and finer...
- Is there an ultimate fine equivalence notion along this line?

The discovery of bisimulation in modal logic:

- Matching the logical equivalence of basic modal logic.
- Homomorphism: too weak
- Strong homomorphism: still too weak
- Surjective strong homomorphism: too strong
- Change the surjectivity to Zig and Zag: bounded morphism. Still too strong and not an equivalence!
- Function to relation: bisimulation. Is it alright?

from homomorphism to bounded morphism

Idea: change the second *iff* condition to *forward and backward* conditions.

Definition (Bounded morphism (p-morphism))

A bounded morphism from $\mathcal{M}$ to $\mathcal{N}$ is a function $f : W_{\mathcal{M}} \to W_{\mathcal{N}}$ such that:

- For any proposition letter $p \in \mathsf{P}$: $p \in V(w) \iff p \in V(f(w))$;
- For all a: if $w \stackrel{a}{\to} w'$ in $\mathcal{M}$ then $f(w) \stackrel{a}{\to} f(w')$ in $\mathcal{N}$.
- For all a: if $f(w) \stackrel{a}{\to} v'$ in $\mathcal{N}$ then there is a w' in $\mathcal{M}$ such that $f(w') = v'$ and $w \stackrel{a}{\to} w'$ in $\mathcal{M}$.

Bisimulation

Definition (Bisimulation (unary similarity types))

A *non-empty* binary relation Z between the domains of two models $\mathcal{M}$ and $\mathcal{N}$ is called a *bisimulation* iff whenever $(w, v) \in Z$ the following conditions are satisfied:

Invariance For any $p \in \mathbf{P} : p \in V_{\mathcal{M}}(w) \iff p \in V_{\mathcal{N}}(v)$.

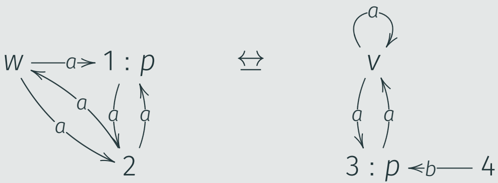
Zig For all a: if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there exists a v' in $\mathcal{N}$ such that $v \xrightarrow{a} v'$ and $w'Zv'$.

Zag For all a: if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there exists a w' in $\mathcal{M}$ such that $w \xrightarrow{a} w'$ and $w'Zv'$.

$(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ are said to be *bisimilar* $(\mathcal{M}, w \leftrightarroweq \mathcal{N}, v)$ if there is a bisimulation Z between them such that $(w, v) \in Z$. We say a bisimulation Z is *total*, if every world in any model is linked by Z to some world in the other model (notation: $\mathcal{M} \leftrightarroweq \mathcal{N}$).

Bisimulation

Example



$Z = \{(w, v), (2, v), (1, 3)\}$.

Z is not total.

About the notation

A *bisimulation* between $\mathcal{M}, \mathcal{N}$ is a relation $Z \subseteq W_{\mathcal{M}} \times W_{\mathcal{N}}$. Is it an *equivalence relation* by definition?

Bisimilarity ($\leftrightarroweq$) is the *equivalence relation* between pointed models (why?) such that $\mathcal{M}, w \leftrightarroweq \mathcal{N}, v$ iff *there is* a bisimulation between $\mathcal{M}$ and $\mathcal{N}$ linking w and v. When the two models are clear from the context we also write $w \leftrightarroweq v$ for $\mathcal{M}, w \leftrightarroweq \mathcal{N}, v$.

However, in practice we often talk about *bisimilarity* by using the word *bisimulation*, e.g., when we say bisimulation is finer than trace equivalence we actually mean: $\leftrightarroweq \subseteq \approx_{trace}$; modal logic is invariant under bisimulation: $\leftrightarroweq \subseteq \equiv_{\mathtt{ML}}$.

Another reason for this abuse of terminology is that bisimilarity can be viewed as the union of all the bisimulations (or say the largest bisimulation). We will come back to this later.

What about this definition of bisimilarity directly?

$\mathcal{M}, w \leftrightarroweq^{\circ} \mathcal{N}, v$ **iff**:

Invariance For any $p \in \mathsf{P} : p \in V_{\mathcal{M}}(w) \iff p \in V_{\mathcal{N}}(v)$.

Zig For all a: if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there exists a v' in $\mathcal{N}$ such that $v \xrightarrow{a} v'$ and $\mathcal{M}, w' \leftrightarroweq^{\circ} \mathcal{N}, v'$.

Zag For all a: if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there exists an w' in $\mathcal{M}$ such that $w \xrightarrow{a} w'$ and $\mathcal{M}, w' \leftrightarroweq^{\circ} \mathcal{N}, v'$.

$? \leftrightarroweq^{\circ} ?$

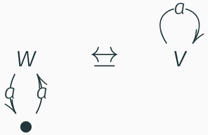
The original definition (massaged version for defining bisimilarity between pointed models directly):

$\mathcal{M}, w_0 \leftrightarroweq \mathcal{N}, v_0$ iff there exists a non-empty $Z \subseteq W_{\mathcal{M}} \times W_{\mathcal{N}}$ such that $(w_0, v_0) \in Z$ and for any $(w, v) \in Z$

Invariance For any $p \in \mathsf{P} : p \in V_{\mathcal{M}}(w) \iff p \in V_{\mathcal{N}}(v)$.

Zig For all a: if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there exists a v' in $\mathcal{N}$ such that $v \xrightarrow{a} v'$ and $w'Zv'$.

Zag For all a: if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there exists a w' in $\mathcal{M}$ such that $w \xrightarrow{a} w'$ and $w'Zv'$.



There is no circularity here.

Features of bisimilarity

Bisimilarity (within one model) can be viewed as the union of all the possible bisimulations: the maximal bisimulation, and it is indeed an equivalence relation.

- locality
- not an inductive definition but a *coinductive* one
- coinductive proof method

Cold jokes:

A comathematician is a device for turning cotheorems into ffee.

Every nut is a coconut.

`https://math.stackexchange.com/questions/900390/joke-explanation-a-comathematician-is-a-device-for-turning-cothe`

Induction and coinduction

Induction:

- construction from the basis
- the least solution $X^\star$ of an inequality: $f(X) \subseteq X$ (closure property)
- proof method: $f(Y) \subseteq Y \implies X^\star \subseteq Y$: $X^\star$ has property Y.

Coinduction:

- destruction from the whole
- the greatest solution $X^\star$ of an inequality: $X \subseteq f(X)$
- proof method: $Y \subseteq f(Y) \implies Y \subseteq X^\star$: Y has property $X^\star$.

Actually, coinduction can be viewed as induction on the complement, let $Y = \overline{X}$, and let $g(Y) = \overline{f(\overline{Y})}$, then $X \subseteq f(X)$ can be reformalized as $g(Y) \subseteq Y$. The least solution of Y is the greatest solution of X for $X \subseteq f(X)$.

Examples

Defining the set of points that are reachable from a point w_0 in a model as the least solution of X such that $f(X) \subseteq X$ where:

$$f(X) = \{w \mid w_0 \rightarrow w\} \cup \{w \mid \exists v \in X : v \rightarrow w\}$$

Defining the set of points that have infinite descending chains in a model as the greatest solution of X such that $X \subseteq f(X)$ where:

$$f(X) = \{w \mid \exists v \in X : w \rightarrow v\}$$

A modal μ-calculus formula: $\nu X.\Diamond X$ (where ν is the greatest fixed point operator) can thus define all the points that have infinite descending chains.

Induction and coinduction

To guarantee those least/greatest solutions do exist:

Lemma

Let $\mu = \bigcap\{X \mid f(X) \subseteq X\}$ and $\nu = \bigcup\{X \mid X \subseteq f(X)\}$. If f is a order-preserving (monotone) function ($X \subseteq Y \implies f(X) \subseteq f(Y)$) then $f(\mu) \subseteq \mu$ and $\nu \subseteq f(\nu)$.

Based on this we can show:

Theorem (Knaster-Tarski, on power sets over W)

If f is a monotone function on subsets of U: $\mathcal{P}(W) \to \mathcal{P}(W)$ then

- *μ is the least fixed point of f.*
- *ν is the greatest fixed point of f.*

Moreover, we can reach μ and ν by (transfinite) iteration of f from $\emptyset$ or W respectively (Kleene's fixed point theorem.)

Fixed-point perspective

Fix a model $\mathcal{M}$. Let us consider the function $f_{\leftrightarroweq} : \mathcal{P}(W_\mathcal{M} \times W_\mathcal{M}) \to \mathcal{P}(W_\mathcal{M} \times W_\mathcal{M})$. $f_{\leftrightarroweq}(Z)$ is defined as the set of pairs (w, v) such that:

Invariance For any $p \in \mathsf{P} : p \in V_\mathcal{M}(w) \iff p \in V_\mathcal{M}(v)$.

Zig if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there exists a v' in $\mathcal{M}$ such that $v \xrightarrow{a} v'$ and $w'Zv'$.

Zag if $v \xrightarrow{a} v'$ in $\mathcal{M}$ then there exists an v' in $\mathcal{M}$ such that $w \xrightarrow{a} w'$ and $w'Zv'$.

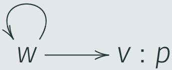
Bisimilarity within this model $(\{(w, v) \mid \mathcal{M}, w \leftrightarroweq \mathcal{M}, v\})$ is the greatest fixed point of $f_{\leftrightarroweq}$ (the greatest solution of the inequation $Z \subseteq f_{\leftrightarroweq}(Z)$. Note that each (non-empty) solution is simply a bisimulation, the union of all the bismulations is the greatest fixed point). Existence of the fixed point is guaranteed by Knaster-Tarski fixed-point theorem ($f_{\leftrightarroweq}$ is monotone). What is the least fixed point?

To find bisimilarity in finite models

The fixed point view also tells us a way to find bisimilarity in practice (Kleene fixed point theorem):

$$Z_0 = W \times W, Z_1 = f_{\underline{\leftrightarrow}}(Z_0), Z_2 = f_{\underline{\leftrightarrow}}(Z_1), \ldots$$

Example (we only focus on the cross-model pairs)



$\{(w, s), (w, t), (w, r), (v, s), (v, t), (v, r)\}$
$\{(w, s), (w, t), (v, r)\}$
$\{(w, t), (v, r)\}$
$\{(v, r)\}$

Approximations of bisimilarity

If we start from $\{(w, v) \mid V(w) = V(v)\}$ then we will have the $\underline{\leftrightarrow}_k$ approximations of $\underline{\leftrightarrow}$.

Definition (n-bisimilarity $\underline{\leftrightarrow}_n$)

1. for any $\mathcal{M}, w, \mathcal{N}, v$: $\mathcal{M}, w \underline{\leftrightarrow}_0 \mathcal{N}, v$ iff $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$
2. $\mathcal{M}, w \underline{\leftrightarrow}_{n+1} \mathcal{N}, v$ if:
    - $\mathcal{M}, w \underline{\leftrightarrow}_n \mathcal{N}, v$
    - for any a, if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there is a v' such that $v \xrightarrow{a} v'$ and $\mathcal{M}, w' \underline{\leftrightarrow}_n \mathcal{N}, v'$
    - for any a, if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there is a w' such that $w \xrightarrow{a} w'$ and $\mathcal{M}, w' \underline{\leftrightarrow}_n \mathcal{N}, v'$

(Compare this to the definition 2.30 in the blue book). Can we replace the first condition ($\mathcal{M}, w \underline{\leftrightarrow}_n \mathcal{N}, v$) by $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$? Let $\underline{\leftrightarrow}_\omega = \bigcap_{n \geq 0} \underline{\leftrightarrow}_n$ we **will** show $\underline{\leftrightarrow} \neq \underline{\leftrightarrow}_\omega$ via games! $\underline{\leftrightarrow}_\omega$ is not always the greatest fixed point for $f_{\underline{\leftrightarrow}}(Z)$.