Advanced Modal Logic II

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Advanced Modal Logic (2024 Spring)

Last time

- Modal logic was discussed already by Aristotle.
- We focus on *propositional* modal logic.
- Modal logic is non-classical.
- Modal logic is the logic of “necessity” and “possibility”.
- Modal logic is obtained from propositional logic by adding a modality operator $\Box$.
- Modal logic is a fragment of first-order/2nd order logic.
- Modal logic is useful in many application areas in CS.
- Modal logic is in PSPACE(a complexity class).

Taking concepts from the meta language to the object language. But not in a way of encoding as in mathematical theories...

Modal logic is widely used to formalize philosophical theories.

To cover in this course

- **model theory of modal logic**
- “frame theory” of modal logic
- proof systems and completeness
- algebraic modal logic

Language, models & semantics

Language, models & semantics

Language, model, semantics

From model-theoretical point of view, a logic framework is a triple: $\langle \textit{Language}, \textit{Models}, \textit{Semantics} \rangle \quad (\mathbf{L}, \mathbb{C}, \vDash)$.

The basic modal language $\mathbf{ML}(\mathbf{P}, \Box)$ based on a countable set $\mathbf{P}$ of proposition letters is defined as follows in the BNF form:

$$\varphi \quad ::= \quad \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

where $p \in \mathbf{P}$. $\Box$ is not a connective nor a predicate.

We define $\bot$ as $\neg\top$, $\varphi \vee \psi$ as $\neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi$ as $\neg(\varphi \wedge \neg\psi)$, and $\Diamond\varphi$ as $\neg\Box\neg\varphi$. (only in the classical setting)

Which modality ($\Box$ or $\Diamond$) is primitive matters in some later technical definitions.

When $\mathbf{P}$ is clear in the context, we use $\mathbf{ML}$ to denote the basic modal language with a single unary modality $\Box$.

Kripke models and semantics: the intuition

A Kripke *model* (relational model) can be roughly pictured as a graph with labelled directed edges and nodes:

- there are some nodes (*possible worlds*, *states*, *situations* or even *objects* etc.), labelled by propositional letters;
- some relations among them labelled by modality.

A *frame* is a model without labels on nodes.

Semantic intuition: φ is *necessarily* true at the current world iff φ is true at all the possible alternatives of the current world.

$\Box\varphi$ holds at a world w iff φ is true at all the *successors* of w.

This semantics is often called Kripke semantics or possible worlds semantics. The “meaning” of one world depends on its connections with others (structuralism).

Kripke frames and models

Talking about model, we often say "a model for *some language*"

Definition (Kripke frame and Kripke model)

A *Kripke frame* for $\mathbf{ML}(\mathsf{P}, \Box)$ is a pair: $\mathcal{F} = \langle W, R \rangle$ where:

- W is a *non-empty* set;
- $R \subseteq W \times W$ is a binary relation on W.

A *Kripke model* $\mathcal{M}$ for $\mathbf{ML}(\mathsf{P}, \Box)$ is a pair $\langle \mathcal{F}, V \rangle$ where $\mathcal{F}$ is a frame and $V : W \to 2^{\mathsf{P}}$ is a *valuation function*. A *pointed Kripke model* $(\mathcal{M}, w)$ is a Kripke model with a *designated* point w in the set of states of $\mathcal{M}$.

2^{P} is a set of functions from P to $\{0, 1\}$, and can be viewed as the power set of P (modulo isomorphism).

It is also common to define $V : \mathsf{P} \to 2^{W}$ in the literature.

Kripke Semantics

Recall the language $\mathsf{ML}(\mathsf{P}, \Box)$:

$$\varphi \quad ::= \quad \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

The satisfaction relation $\vDash$ is defined over pointed Kripke models for the same language as follows:

Kripke Semantics

$$\begin{array}{rcl}
\mathcal{M}, w \vDash \top & & \text{always} \\
\mathcal{M}, w \vDash p & \Leftrightarrow & p \in V(w) \\
\mathcal{M}, w \vDash \neg\varphi & \Leftrightarrow & \mathcal{M}, w \nvDash \varphi \\
\mathcal{M}, w \vDash (\varphi \wedge \psi) & \Leftrightarrow & \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}, w \vDash \psi \\
\mathcal{M}, w \vDash \Box\varphi & \Leftrightarrow & \text{for all } v : \text{ if } wRv \text{ then } \mathcal{M}, v \vDash \varphi
\end{array}$$

If $\mathcal{M}, w \vDash \varphi$, we say φ is *true* or *satisfiable* at w in $\mathcal{M}$.

More generally...

There can be multiple modalities, and a modality can have an arity greater than one.

The modal language based on a set P of proposition letters and a *modal similarity type* $\tau = (\mathsf{O}, \rho)$ is as follows $\mathsf{ML}(\mathsf{P}, \tau)$:

$$\varphi \quad ::= \quad \top \mid p \mid (\varphi \wedge \varphi) \mid \neg\varphi \mid \nabla(\underbrace{\varphi \ldots \varphi}_{\rho(\nabla)})$$

where $p \in \mathsf{P}, \nabla \in \mathsf{O}$ and $\rho : \mathsf{O} \to \mathbb{N}$ (What if $\rho(\nabla)$ is 0?)
We define $\Delta(\varphi_1 \ldots \varphi_{\rho(\nabla)})$ as $\neg\nabla(\neg\varphi_1 \ldots \neg\varphi_{\rho(\nabla)})$.

Kripke frames and models, more generally

Definition (Kripke frame and Kripke model)

A *Kripke frame* for $\mathsf{ML}(\mathsf{P}, (\mathsf{O}, \rho))$ is a pair $\mathcal{F} = \langle W, \{R_\nabla \mid \nabla \in \mathsf{O}\}\rangle$ where:

- W is a *non-empty* set;
- $R_\nabla \subseteq \underbrace{W \times \cdots \times W}_{\rho(\nabla)+1}$ is a $(\rho(\nabla) + 1)$-ary relation on W.

Since $\mathcal{F}$ is not about P we also call such $\mathcal{F}$ a τ-frame given a similarity type τ. A *Kripke model* $\mathcal{M}$ for $\mathsf{ML}(\mathsf{P}, \tau)$ is a pair $\langle \mathcal{F}, V\rangle$ where $\mathcal{F}$ is a τ-frame and $V : W \to 2^{\mathsf{P}}$ is a *valuation function*. A *pointed Kripke model* $(\mathcal{M}, w)$ is a Kripke model with a *designated* point w in the set of states of $\mathcal{M}$.

Kripke Semantics

Recall the language $\mathsf{ML}(\mathsf{P}, (\mathsf{O}, \rho))$:

$$\varphi \quad ::= \quad \top \mid p \mid (\varphi \wedge \varphi) \mid \neg\varphi \mid \nabla(\underbrace{\varphi \ldots \varphi}_{\rho(\nabla)})$$

Pay attention to the quantifiers in the last clause:

Kripke Semantics ($\rho(\nabla) > 0$)

$$\begin{aligned}
\mathcal{M}, w \vDash \top &\quad\Leftrightarrow\quad \text{always} \\
\mathcal{M}, w \vDash p &\quad\Leftrightarrow\quad p \in V(w) \\
\mathcal{M}, w \vDash \neg\varphi &\quad\Leftrightarrow\quad \mathcal{M}, w \nvDash \varphi \\
\mathcal{M}, w \vDash (\varphi \wedge \psi) &\quad\Leftrightarrow\quad \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}, w \vDash \psi \\
\mathcal{M}, w \vDash \nabla(\varphi_1 \ldots \varphi_{\rho(\nabla)}) &\quad\Leftrightarrow\quad \forall w_1 \ldots w_{\rho(\nabla)} : (\langle w, w_1, \ldots, w_{\rho(\nabla)} \rangle \in R_\nabla \\
&\qquad\qquad \text{implies } \exists i \in [1, \rho(\nabla)] : \mathcal{M}, w_i \vDash \varphi_i)
\end{aligned}$$

Why do we define it like this?

Kripke models of unary similarity types

Given $\tau = (\mathsf{O}, \rho)$, if for all $\nabla \in \mathsf{O}$: $\rho(\nabla) = 1$ then we call τ a *unary similarity type*. In such cases we often use indexed boxes (e.g., $\Box_a$) to denote modalities and write $w \xrightarrow{a} v$ for $(w, v) \in R_a$ in the frames and models. We say a model is of a unary similarity type if its frame is of a unary similarity type. Given $\mathcal{M}$, we write $W_{\mathcal{M}}$, $R_a^{\mathcal{M}}$ (or $\xrightarrow{a}_{\mathcal{M}}$) and $V_{\mathcal{M}}$ for the corresponding components in $\mathcal{M}$.

$\mathcal{M}, w \models \Box_a \varphi$ *iff* for all $v : w \xrightarrow{a}_{\mathcal{M}} v \implies \mathcal{M}, v \models \varphi$
$\mathcal{M}, w \models \neg \Box_a \neg \varphi$ *iff* not the case that $(\forall v : w \xrightarrow{a} v \implies \mathcal{M}, v \models \neg \varphi)$
$\mathcal{M}, w \models \Diamond_a \varphi$ *iff* there exists $v : w \xrightarrow{a}_{\mathcal{M}} v$ and $\mathcal{M}, v \models \varphi$

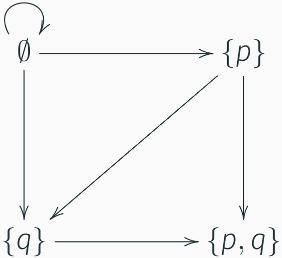
Some properties of Kripke models

A Kripke model $\mathcal{M}$ of a unary similarity type is said to be:

- *finite*, if $W_{\mathcal{M}}$ is finite. (In CS the relations are also required to be finitely representable.)
- *image-finite*, if for any world $w \in W_{\mathcal{M}}$, any $a \in \mathsf{O}$, $\{v \mid w \xrightarrow{a}_{\mathcal{M}} v\}$ is finite.
- *deterministic*, if for any world $w \in W_{\mathcal{M}}$, any $a \in \mathsf{O}$, $w \xrightarrow{a}_{\mathcal{M}} v$ and $w \xrightarrow{a}_{\mathcal{M}} v'$ implies $v = v'$.
- *well-founded*, if there is no infinite upward sequence: $\ldots \xrightarrow{a_2}_{\mathcal{M}} w_2 \xrightarrow{a_1}_{\mathcal{M}} w_1 \xrightarrow{a_0}_{\mathcal{M}} w_0$.
- *acyclic*, if there is no cycle.
- $\mathcal{M}, w$ is a *tree* if $\mathcal{M}$ is acyclic, w can reach (via some nodes) all the other nodes, and each non-w node has a unique predecessor.

An example

Consider the following Kripke model $\mathcal{M}$ of the basic similarity type ($\tau = (\{\Box\}, \rho)$ and $\rho(\Box) = 1$):



Where is $\Box\Diamond p$ true? Where is $\Box\Diamond p \wedge \Diamond\Box p$ true?

Model checking problems

Two model checking problems:

1. Local model checking: testing whether $\mathcal{M}, w \vDash \varphi$;
2. Global model checking: compute the set $\{w \in W_{\mathcal{M}} \mid \mathcal{M}, w \vDash \varphi\}$

Let $l_R(X) = \{w \in W_{\mathcal{M}} \mid \forall v : w \rightarrow_{\mathcal{M}} v \implies v \in X\}$, we can define the *extension* of formulas in $\mathcal{M}$:

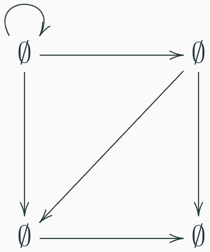
$$
\begin{array}{rcl}
[\![\top]\!]^{\mathcal{M}} & = & W_{\mathcal{M}} \\
[\![\neg\varphi]\!]^{\mathcal{M}} & = & W \backslash [\![\varphi]\!]^{\mathcal{M}} \\
[\![\Box\varphi]\!]^{\mathcal{M}} & = & l_R[\![\varphi]\!]^{\mathcal{M}}
\end{array}
\qquad
\begin{array}{rcl}
[\![p]\!]^{\mathcal{M}} & = & \{w \mid p \in V(w)\} \\
[\![(\varphi \wedge \psi)]\!]^{\mathcal{M}} & = & [\![\varphi]\!]^{\mathcal{M}} \cap [\![\psi]\!]^{\mathcal{M}}
\end{array}
$$

An algorithm for global model checking φ on $\mathcal{M}$: labelling the states of $\mathcal{M}$ by the sub-formulas of φ (according to their complexity) that are true at each state. What about matrix representations? The problem is non-trivial in practice since the number of possibilities can be huge…

Model checking temporal logics (2007 Turing Award).

An example

Consider the following Kripke model $\mathcal{M}$ of the basic similarity type ($\tau = \langle \{\Box\}, \rho \rangle$ and $\rho(\Box) = 1$):



Where is $\Diamond\Box p$ true? Where is $\Box\Diamond p \wedge \Diamond\Box p$ true? Can we characterize each state by a modal formula?

Satisfiability and Validity

Modal formulas can talk about models or frames at local or global levels, this induces 4 cases:

- φ is *satisfiable* at $\mathcal{M}, w$ if $\mathcal{M}, w \vDash \varphi$.
- φ is *valid* in $\mathcal{M}$ ($\mathcal{M} \vDash \varphi$) if for all w in $\mathcal{M}$: $\mathcal{M}, w \vDash \varphi$.
- φ is *valid* in a *pointed* frame $\mathcal{F}, w$ ($\mathcal{F}, w \vDash \varphi$) if for any model based on $\mathcal{F}, w$: $\mathcal{M}, w \vDash \varphi$.
- φ is *valid* in a frame $\mathcal{F}$ ($\mathcal{F} \vDash \varphi$) if for any model based on $\mathcal{F}$: $\mathcal{M} \vDash \varphi$.

	model	frame
local	$\mathcal{M}, w \vDash \varphi$	$\mathcal{F}, w \vDash \varphi$
global	$\mathcal{M} \vDash \varphi$	$\mathcal{F} \vDash \varphi$

Similarly, we can define the validity of a formula w.r.t. classes of models or frames.

Further reading

Max Cresswell: Modal Logic before Kripke. Organon F 26(3),(2019)

B. Jack Copeland: The Genesis of Possible Worlds Semantics. J. Philosophical Logic 31(2): 99-137 (2002)

B. Jack Copeland: Meredith, Prior, and the History of Possible Worlds Semantics. Synthese 150(3): 373-397 (2006)