



# Epistemic Logic XIII

Bundled fragments of first-order modal logic

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Recap

A Logic of Mention-Some (and All)

## Recap

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# The logic tool for knowing-wh

knowledge-that — propositional modal logic  
knowledge-wh — first-order modal logic

In *Meaning and Necessity* (1947), Carnap remarked:

*Any system of modal logic without quantification is of interest only as a basis for a wider system including quantification. If such a wider system were found to be impossible, logicians would probably abandon modal logic entirely.*

However, it seems that history went exactly the other way around.

# Many things can be done in first-order modal logic

First-order modal logic is **infamous** for:

- issues in the semantics
- *quantifying-in* and substitution
- ambiguity: *de re* vs. *de dicto*
- incompleteness
- lack of Craig's interpolation
- undecidability (hard to find useful decidable fragments)
- ....

At the same time, propositional modal logic is **too** successful...

## Beyond knowing that: starting point

As philosophical logicians, we design specific-purpose languages to stay at the appropriate abstraction level to highlight the concepts in concern.

Instead of using the full language of first-order modal logic, we can use some well-behaved *fragments* of it to focus on what we really care but no more.

Can we repeat the success of propositional modal logic by a systematic approach to know-wh?

- simple language
- intuitive semantics
- useful models
- balanced expressive power and complexity...

## The minimalist's “bundle” approach [Wang18]

- take a know-wh construction as a **single** modality (a “bundle”), e.g., pack  $\exists x \Box (Mary \approx x)$  into ***Kwho*** *Mary*
- the use of quantifiers is restricted (recall the secret of success of propositional modal logic).
- natural and succinct to express the desired properties, e.g., *I know that you know what the password is but I do not know the password.*
- capture the essence of the relevant reasoning by axioms.
- **lead to new decidable fragments of first-order modal logic.**
- lead to intuitive understanding of non-classical logics.
- stay (technically) neutral for certain philosophical issues.

## For each know-wh: the general steps

- focus on some (logically) interesting interpretation
- give natural semantics guided by the first-order modal formulation and linguistic/philosophical theories;
- axiomatize logics with (combinations of) new operators;
- simplify the semantics while keeping the validities;
- capture the expressivity via notions of bisimulation;
- dynamify those logics with new updates of knowledge;
- automate the inferences based on decidability;
- come back to philosophy and linguistics with new insights and questions.

## Beyond knowing that: (technical) difficulties

- (apparently) not normal:
  - $\not\vdash Kw(p \rightarrow q) \wedge Kw p \rightarrow Kw q$
  - $\not\vdash Kh\varphi \wedge Kh\psi \rightarrow Kh(\varphi \wedge \psi)$
  - $\vdash \varphi \not\Rightarrow \vdash Ky\varphi$
- not strictly weaker:  $\vdash Kw\varphi \leftrightarrow Kw\neg\varphi$ ;
- combinations of quantifiers and modalities, e.g.,  $\exists x\Box\varphi(x)$ ;
- the things that we quantify vary a lot;
- the axioms depend on the special shape of  $\varphi$  as well;
- weak language vs. rich model: hard to axiomatize;
- fragments of FO/SO-modal language: we know little.

# Characteristic feature

How to distinguish the work in this line and other related work in the literature?

Whether it uses a **single** modality for a type of know-wh, instead of breaking it down into quantifiers, normal modalities, questions, predicates and so on.

It also gives us a new “looking glass” to understand the world.

## Some knowing-wh logics we proposed and studied

wh-word	bundle	connection	key ref
whether	$\Box\varphi \vee \Box\neg\varphi$	non-contingency logic	[FWvD14,15]
what	$\exists x\Box(\varphi \rightarrow x \approx c)$	weakly aggregative logic	[WF13,14]
how	$\exists\pi\Box[\pi]\varphi$	game logic, ATL	[Wang15,17]
why	$\exists t\Box(t : \varphi)$	justification logic	[XWS18]

We obtained complete axiomatizations, characterizations of expressive power, and decidability ...

Along the way, we understand better why neighbourhood-like semantics works for various philosophical logics.

# Connections to existing logics and linguistic theories

Classification by question words:

- Knowing whether: non-contingency logic, ignorance logic
- Knowing what: weakly aggregative logic, dependence logic
- Knowing how: game Logic, alternating temporal logic
- Knowing why: (quantified) justification Logic
- Knowing who: (dynamic) termed modal logic

Classification by logical forms:

- *Mention-some*: e.g., *knowing how/why...*  $\exists x \mathcal{K} \varphi(x)$
- *Mention-all* (strongly exhaustive reading): e.g., *I know who came to the party...*  $\forall x (\mathcal{K} \varphi(x) \vee \mathcal{K} \neg \varphi(x))$
- *In-between*: *know-value*  $\exists x (\mathcal{K} c \approx x) \leftrightarrow \forall x (\mathcal{K} c \approx x \vee \mathcal{K} c \not\approx x)$

# Epistemic logic: form one to many

(Routine) research questions:

- Model theory, proof theory, computational complexity
- Group knowledge
- Logical omniscience
- Natural dynamics
- Applications

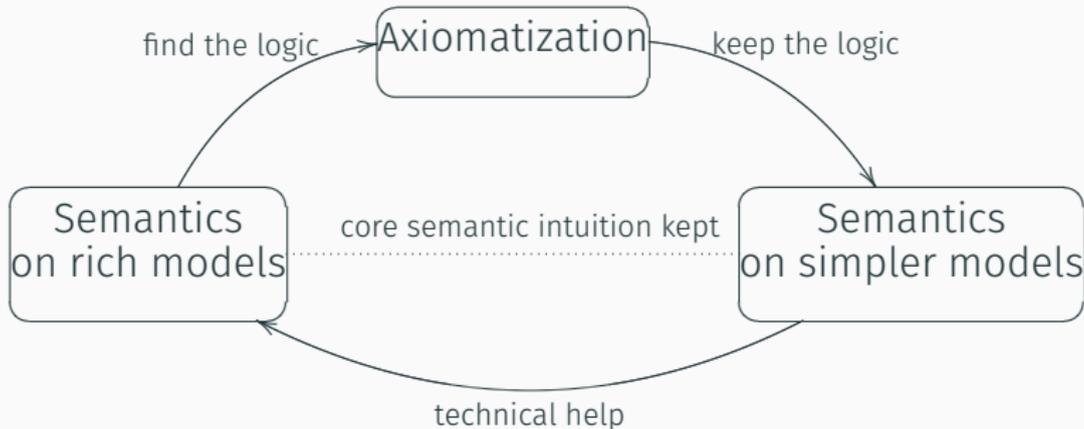
New questions:

- Interactions of different knowledge expressions;
- Simplification of semantics.

# Simplify the semantics while keeping the logic

Common difficulties: weak language vs. rich semantics

To restore the balance between the language and model:



# Disadvantages of those concrete logics

'Disadvantages' from a linguistic point of view:

- Compositionality
- Uniformity
- Expressivity

Disadvantages in terms of knowledge representation:

- Propositional epistemic logic is not really about the *content* of knowledge!

A question: how to explain the decidability of those logics?

# Towards a general new framework

What we are after:

- Expressive enough: covering the essence of those non-standard epistemic logics
- Not too much: sharing most good properties of propositional modal logic

Uniformity, compositionality, expressivity, computability: we want a predicate modal framework like the propositional modal logic

# A new framework for predicate epistemic logic

Inspired by the concrete know-wh logics, we introduce the bundle modalities into the predicate modal language:

- pack  $\exists x\mathcal{K}$  into a *bundle* modality (mention-some)
- pack  $\forall x\Delta$  into a *bundle* modality (mention-all)

You can also come up with your favourite bundles inspired by the categorization of the penex forms for the *classical decision problem*.

We obtain some nice and powerful fragments of first-order modal logic.

## A Logic of Mention-Some (and All)

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## Definition ( $\exists\Box$ -fragment)

Given set of variables  $X$  and set of predicate symbols  $Ps$ ,

$$\varphi ::= P\bar{x} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x\Box\varphi$$

where  $x, y \in X, P \in Ps$ .

$\exists x\Box\varphi$  reads 'I know some  $x$  such that  $\varphi(x)$ '.

$\Box\varphi$  is expressible by  $\exists x\Box\varphi$  if  $x$  does not occur free in  $\varphi$ .

We can add the equality symbol.

- Knowing-wh:  $\exists x \Box \varphi(x)$
- “I know a theorem of which I do not know any proof”:  
 $\exists x \Box \neg \exists y \Box \text{Prove}(y, x)$
- “*a* knows a country which *b* knows its capital”:  
 $\exists x \Box_a \exists y \Box_b \text{Capital}(y, x)$

# First-order Kripke semantics

## Definition (First-order Kripke Model)

An *increasing domain* model  $\mathcal{M} = \langle W, D, \delta, R, \rho \rangle$  where:

$W$  is a non-empty set.

$D$  is a non-empty set.

$R \in 2^{W \times W}$  is a binary relation over  $W$ .

$\delta : W \rightarrow 2^D$  assigns to each  $w \in W$  a *non-empty* local domain s.t.  $wRv$  implies  $\delta(w) \subseteq \delta(v)$  for any  $w, v \in W$ .

$\rho : \text{Ps} \times W \rightarrow \bigcup_{n \in \omega} 2^{D^n}$  such that  $\rho$  assigns each  $n$ -ary predicate on each world an  $n$ -ary relation on  $D$ .

We write  $D_w^{\mathcal{M}}$  for the local domain  $\delta(w)$  in  $\mathcal{M}$ . If  $\delta(w) = \delta(w')$  for all  $w, w'$  then it is called a *constant domain* model.

## Definition ( $\exists\Box$ Semantics)

$\mathcal{M}, w, \sigma \models \exists x\Box\varphi \Leftrightarrow$  there exists an  $a \in D_w^{\mathcal{M}}$  such that  
 $\mathcal{M}, v, \sigma[x \mapsto a] \models \varphi$  for all  $v$  s.t.  $wRv$   
 $\Leftrightarrow$  there exists an  $a \in D_w^{\mathcal{M}}$  such that  
 $\mathcal{M}, w, \sigma[x \mapsto a] \models \Box\varphi$

$\exists\Box$  fragment is indeed an extension of ML:

$\models \Box\varphi \leftrightarrow \exists x\Box\varphi$  (given  $x$  does not appear free in  $\varphi$ ).

A formula  $\varphi$  is *satisfiable* if there is an increasing domain pointed model  $\mathcal{M}, w$  and an assignment  $\sigma$  such that  $\mathcal{M}, w, \sigma \models \varphi$  and  $\sigma(x) \in D_w^{\mathcal{M}}$  for all  $x \in X$ .

## $\exists\Box$ -Bisimulation (inspired by monotonic and obj-world bis)

Given  $\mathcal{M}$  and  $\mathcal{N}$ , non-empty  $Z \subseteq (W_{\mathcal{M}} \times D_{\mathcal{M}}^*) \times (W_{\mathcal{N}} \times D_{\mathcal{N}}^*)$  is called an  $\exists\Box$ -bisimulation, if for every  $((w, \bar{a}), (v, \bar{b})) \in Z$  such that  $|\bar{a}| = |\bar{b}|$  the following holds (we write  $w\bar{a}$  for  $(w, \bar{a})$ ):

PISO  $\bar{a}$  and  $\bar{b}$  form a partial isomorphism interpretations of predicates at  $w$  and  $v$  respectively.

$\exists\Box$ Zig For any  $c \in D_w^{\mathcal{M}}$ , there is a  $d \in D_v^{\mathcal{N}}$  such that for any  $v' \in W_{\mathcal{N}}$  if  $vRv'$  then there exists  $w'$  in  $W_{\mathcal{M}}$  such that  $wRw'$  and  $w'\bar{a}cZv'\bar{b}d$ . ( $\forall_{\mathcal{M}}^{\text{object}} \exists_{\mathcal{N}}^{\text{object}} \forall_{\mathcal{N}}^{\text{world}} \exists_{\mathcal{M}}^{\text{world}}$ )

$\exists\Box$ Zag Symmetric to  $\exists\Box$ Zig.

We say  $\mathcal{M}, w\bar{a}$  and  $\mathcal{N}, v\bar{b}$  are  $\exists\Box$ -bisimilar ( $\mathcal{M}, w\bar{a} \leftrightarrow_{\exists\Box} \mathcal{N}, v\bar{b}$ ) if  $|\bar{a}| = |\bar{b}|$  and there is an  $\exists\Box$ -bisimulation linking  $w\bar{a}$  and  $v\bar{b}$ . If there is equality symbol then PISO should respect *identity*.

## Example

Consider the *constant domain* models  $\mathcal{M}$  and  $\mathcal{N}$ :

$$\mathcal{M} : \quad \underline{w} \begin{array}{l} \longrightarrow v : Pa \\ \searrow u : Pb \end{array} \quad \mathcal{N} : \quad \underline{s} \begin{array}{l} \longrightarrow t : Pc \\ \searrow r \end{array}$$

where  $D^{\mathcal{M}} = \{a, b\}$ ,  $D^{\mathcal{N}} = \{c\}$ . Suppose  $P$  is the only unary predicate, we can show that  $\mathcal{M}, w \Leftrightarrow_{\exists\Box} \mathcal{N}, s$  by an  $\exists\Box$ -bisimulation  $Z$ :

$$\{(w, s), (va, tc), (ub, tc), (vb, rc), (ua, rc)\}$$

Note that  $\exists\Box\text{Zig}$  and  $\exists\Box\text{Zag}$  hold trivially for  $w\bar{a}$  and  $v\bar{b}$  if  $w$  and  $v$  *do not* have any successor, based on the fact that local domains are non-empty by definition.

# Limited expressive power

## Theorem

$\mathcal{M}, w\bar{a} \Leftrightarrow_{\exists\Box} \mathcal{N}, v\bar{b}$  then  $\mathcal{M}, w\bar{a} \equiv_{\text{MLMS}\approx} \mathcal{N}, v\bar{b}$ .

## Proposition

$\Box\exists xPx$ ,  $\exists x\Diamond Px$  and  $\Diamond\exists xPx$  are **not** expressible in the  $\exists\Box$ -fragment.

For the undefinability of  $\Box\exists Px$  see the previous example.

For  $\exists x\Diamond Px$ , and  $\Diamond\exists xPx$ , consider:



where  $D^{\mathcal{M}} = \{a, b\}$ ,  $D^{\mathcal{N}} = \{c\}$  as before.

A model  $\mathcal{M}$  is said to be  $\exists\Box$ -saturated, if for any  $w \in W^{\mathcal{M}}$ , and any finite sequence  $\bar{a} \in D_{\mathcal{M}}^*$ :

- $\exists\Box$ -type If for each finite subset  $\Delta$  of a set  $\Gamma(\bar{y}x)$  where  $|\bar{y}| = |\bar{a}|$ ,  $\mathcal{M}, w \models \exists x\Box \bigwedge \Delta[\bar{a}]$ , then there is a  $c \in D_w^{\mathcal{M}}$  such that  $\mathcal{M}, w \models \Box\varphi[\bar{a}c]$  for all  $\varphi \in \Gamma$ , where  $x$  is assigned  $c$ .
- $\Diamond$ -type If for each finite subset  $\Delta$  of  $\Gamma(\bar{x})$  such that  $|\bar{x}| = |\bar{a}|$ ,  $\mathcal{M}, w \models \Diamond \bigwedge \Delta[\bar{a}]$ , then there is a  $v$  such that  $wRv$  and  $\mathcal{M}, v \models \varphi[\bar{a}]$  for each  $\varphi \in \Gamma$ .

### Theorem

For  $\exists\Box$ -saturated models  $\mathcal{M}, \mathcal{N}$  and  $|\bar{a}| = |\bar{b}|$ :  
 $\mathcal{M}, w\bar{a} \leftrightarrow_{\exists\Box} \mathcal{N}, v\bar{b} \Leftrightarrow \mathcal{M}, w\bar{a} \equiv_{\text{MLMS}\approx} \mathcal{N}, v\bar{b}$

### Theorem (Wang TARK17)

A first-order modal formula is equivalent to a formula in the  $\exists\Box$ -fragment iff it is invariant under  $\exists\Box$ -bisimulation.

# A complete epistemic logic over S5 models SMLMS

Over S5 (constant-domain) models, **MLMS** is very powerful, it can also express *mention-all* by  $\forall x \diamond (\Box \varphi \vee \Box \neg \varphi)$  (also  $\forall x \Box \varphi$  by  $\forall x \diamond \Box \varphi$ ).

## Axioms

**TAUT** all axioms of propositional logic

**DISTK**  $\Box(\varphi \rightarrow \psi) \rightarrow \Box \varphi \rightarrow \Box \psi$

**T**  $\Box \varphi \rightarrow \varphi$

**4MS**  $\exists x \Box \varphi \rightarrow \Box \exists x \Box \varphi$

**5MS**  $\neg \exists x \Box \varphi \rightarrow \Box \neg \exists x \Box \varphi$

**KtoMS**  $\Box(\varphi[y/x]) \rightarrow \exists x \Box \varphi$  (admissible  $\varphi[y/x]$ )

**MStoK**  $\exists x \Box \varphi \rightarrow \Box \varphi$  (if  $x \notin FV(\varphi)$ )

**MStoMSK**  $\exists x \Box \varphi \rightarrow \exists x \Box \Box \varphi$

**KT**  $\Box \top$

## Rules:

**MP**  $\frac{\varphi, \varphi \rightarrow \psi}{\psi}$

**MONOMS**  $\frac{\vdash \varphi \rightarrow \psi}{\vdash \exists x \Box \varphi \rightarrow \exists x \Box \psi}$

To treat the equality (if we introduce it), we also need **ID** :  $x \approx x$  and **SUBID** :  $x \approx y \rightarrow (\varphi \rightarrow \psi)$ . We can derive **KEQ** :  $x \approx y \rightarrow \Box(x \approx y)$  and **KNEQ** :  $x \not\approx y \rightarrow \Box(x \not\approx y)$ .

We can axiomatize the logic over arbitrary models without

**T. 4MS. 5MS. MStoMSK.**

# Compare with the know-how logic

TAUT

all axioms of propositional logic

MP

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi}$$

DISTK

$$\mathcal{K}p \wedge \mathcal{K}(p \rightarrow q) \rightarrow \mathcal{K}q$$

NECK

$$\frac{\psi}{\varphi}$$

T

$$\mathcal{K}p \rightarrow p$$

EQREPKh

$$\frac{\mathcal{K}\varphi \quad \varphi \rightarrow \psi}{\mathcal{K}h\varphi \rightarrow \mathcal{K}h\psi}$$

4

$$\mathcal{K}p \rightarrow \mathcal{K}\mathcal{K}p$$

SUB

$$\frac{\varphi(p)}{\varphi[\psi/p]}$$

5

$$\neg\mathcal{K}p \rightarrow \mathcal{K}\neg\mathcal{K}p$$

AxKtoKh

$$\mathcal{K}p \rightarrow \mathcal{K}hp$$

AxKhtoKhK

$$\mathcal{K}hp \rightarrow \mathcal{K}h\mathcal{K}p$$

AxKhtoKKh

$$\mathcal{K}hp \rightarrow \mathcal{K}\mathcal{K}hp$$

AxKhKh

$$\mathcal{K}h\mathcal{K}hp \rightarrow \mathcal{K}hp$$

AxKhbot

$$\neg\mathcal{K}h\perp$$

# Completeness proof

## Definition

A set of  $\text{MLMS}^+$  formulas has  $\exists$ -property if for each  $\exists x \Box \varphi \in \text{MLMS}^+$  it contains a “witness” formula  $\exists x \Box \varphi \rightarrow \Box \varphi[y/x]$  for some  $y \in X^+$  where  $\varphi[y/x]$  is admissible.

## Definition (Canonical model)

The canonical model is a tuple  $\langle W^c, D^c, \sim^c, \rho^c \rangle$  where:

- $W^c$  is the set of maximal  $\text{SMLMS}^+$ -consistent sets with  $\exists$ -property,
- $D^c = X^+$ ,
- $s \sim^c t$  iff  $\Box(s) \subseteq t$  where  $\Box(s) := \{\varphi \mid \Box \varphi \in s\}$ ,
- $\bar{x} \in \rho^c(P, s)$  iff  $P\bar{x} \in s$ .

It is routine to show that  $\sim^c$  is an equivalence relation

# Completeness proof

## Lemma

If  $\Box\psi \notin s \in W^c$  then there exists a  $t \in W^c$  such that  $s \sim^c t$  and  $\neg\psi \in t$ .

The witnesses for  $\exists\Box$  formulas can be added by using:

$$\vdash_{\text{SMLMS}} (\exists x\Box\varphi \rightarrow \Box\psi) \rightarrow \Box(\exists x\Box\varphi \rightarrow \Box\psi).$$

## Lemma

Let  $\sigma^*$  be the assignment such that  $\sigma^*(x) = x$  for all  $x \in X^+$ .  
For any  $\varphi \in \mathbf{MLMS}^+$ , any  $s \in W^c$ :

$$\mathcal{M}^c, s, \sigma^* \vDash \varphi \Leftrightarrow \varphi \in S$$

Each SMLMS consistent set can be extended to an SMLMS<sup>+</sup> consistent set.

## What about decidability?

The situation for first-order modal logic is hopeless. Simply putting a decidable fragment of first-order logic plus a modality does not work at all.

Language	Decidability	Ref
$P^1$	undecidable	[Kripke 62]
$x, y, p, P^1$	undecidable	[Gabbay 93]
$x, y, \Box_i, \text{single } P^1$	undecidable	[Rybakov & Shkatov 17]

The decidable fragments are rare (only one  $x$  in  $\Box$ ). Most of the propositional know-wh logics are in the one variable fragment.

Language	Decidability	Ref
single $x$	decidable	[Seegerberg 73]
$x, y / P^1 / GF, \Box_i(x)$	decidable	[Wolter & Zakharyashev 01]

## Tableaux (can be viewed as a satisfiability game)

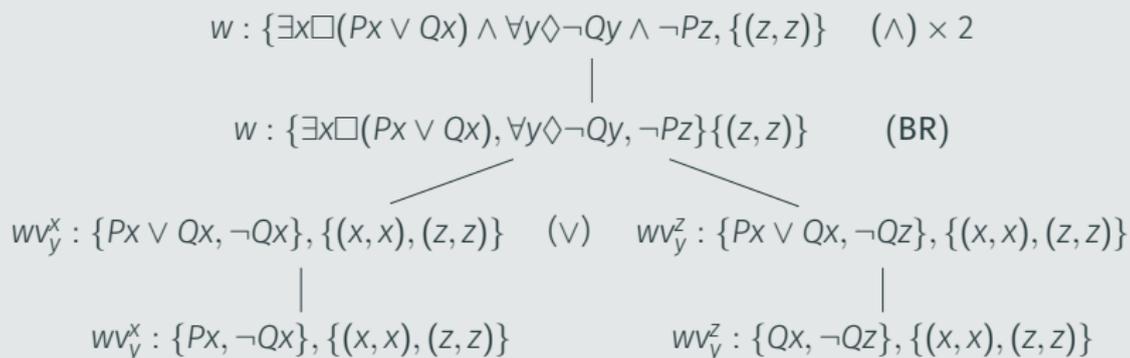
We start from negated normal form (and require some “cleanness”):

$$\varphi ::= P\bar{x} \mid \neg P\bar{x} \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \exists x \square \varphi \mid \forall x \diamond \varphi$$

$\frac{w : \varphi_1 \vee \varphi_2, \Gamma, \sigma}{w : \varphi_1, \Gamma, \sigma \mid w : \varphi_2, \Gamma, \sigma} (\vee) \qquad \frac{w : \varphi_1 \wedge \varphi_2, \Gamma, \sigma}{w : \varphi_1, \varphi_2, \Gamma, \sigma} (\wedge)$
<p>Given <math>n \geq 0, m \geq 1</math>:</p> $\frac{w : \square^{x_1} \varphi_1, \dots, \square^{x_n} \varphi_n, \diamond^{y_1} \psi_1, \dots, \diamond^{y_m} \psi_m, l_1 \dots l_k, \sigma}{\{(wv_{y_i}^y : \{\varphi_j \mid 1 \leq j \leq n\}, \psi_i[y/y_i], \sigma') \mid y \in \text{Dom}(\sigma'), i \in [1, m]\}} (\text{BR})$
<p>Given <math>n \geq 1, k \geq 0</math>:</p> $\frac{w : \square^{x_1} \varphi_1, \dots, \square^{x_n} \varphi_n, l_1 \dots l_k, \sigma}{w : l_1 \dots l_k, \sigma} (\text{END})$

where  $\sigma' = \sigma \cup \{(x_j, x_j) \mid j \in [1, n]\}$  and  $l_k \in \text{lit}$  (the literals),  $\square^x$  is abbreviation of  $\exists x \square$ ,  $\diamond^x$  denotes  $\forall x \diamond$

# An example



### Theorem (Wang TARK17)

*A formula  $\varphi$  in the  $\exists\Box$  fragment is satisfiable iff its NNF has an open tableau.*

### Theorem (Wang TARK17)

*A formula  $\varphi$  in the  $\exists\Box$  fragment is satisfiable over arbitrary increasing domain models then it has a finite tree model whose depth is linearly bound by the length of  $\varphi$ .*

### Corollary (Wang TARK17)

*Satisfiability checking of  $\exists\Box$  fragment over arbitrary increasing domain is PSPACE-complete.*

The  $\exists\Box$  fragment behaves like the basic propositional modal logic but much more powerful.

Moreover, we can show that:

**Theorem (Padmanabha, Ramanujam, Wang FSTTCS18)**

*The  $\exists\forall$ -fragment is decidable over arbitrary constant domain models.*

Actually we can show that:

**Theorem (Padmanabha, Ramanujam, Wang FSTTCS18)**

*The  $\exists\forall$ -fragment cannot distinguish increasing domain and constant domain models. The logic is exactly the same over constant domain models or increasing domain models.*

*The meaning of the world is the separation of wish and fact.*

— Gödel

- $\exists\Box$  fragment is **undecidable** over S5 models: replacing each quantifier in a first-order formula in the prenex form by  $\exists x\Box$  or  $\forall x\Diamond\Box$  respectively qua satisfiability
- $\forall\Box$  fragment with two unary predicates is **undecidable** over constant domain models: use  $\Diamond(P(x) \wedge Q(y))$  to encode the binary predicate, and use  $\forall z_1\Box \forall z_2\Box (\Diamond^n \Diamond (P(z_1) \wedge Q(z_2)) \rightarrow \Box^n \Diamond(P(z_1) \wedge Q(z_2)))$  to force uniformity of evaluation.

It is not as robust as propositional modal logic: we are at the edge of first-order expressivity.

## What about other bundled fragments?

What we know about the decidability of equality-free bundled fragments [Wang 17, Padmanabha, Ramanujam, Wang18]:

Language	Model	Decidability	Remark
$\forall \square, P^1$	Const.	undecidable	
$\square \forall, P^1$	Const.	undecidable	
$\exists \square$	Const.	decidable	PSPACE-complete
$\square \exists$	Const.	?	$\square \exists \exists$ undecidable (Liu 19)

$\exists \square$  over constant model models actually has the same logic as it over increasing domain models. We can show decidability in two ways: turning an increasing domain model into an equivalent constant domain model, or give a bound on the number of necessary fresh variables to be introduced.

Further directions:

- What about adding  $\approx$  and constant symbols (for decidability)?
- Which frame conditions can be added while keeping the decidability.
- Other bundles...
- Model/proof theoretical aspects.

Recall that  $\Delta$  is the knowing whether operator.

## Definition (The $\forall\Delta$ -fragment)

Given  $X$  and  $P$ s, the fragment  $\forall\Delta$  is defined as:

$$\varphi ::= P\bar{x} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Delta\varphi \mid \forall x\Delta\varphi$$

The semantics for  $\Delta$  is as usual.

## Axioms

TAUT all axioms of propositional logic

KwCon  $\Delta\varphi \wedge \Delta\psi \rightarrow \Delta(\varphi \wedge \psi)$

KwDis  $\Delta\varphi \rightarrow \Delta(\varphi \rightarrow \psi) \vee \Delta(\neg\varphi \rightarrow \chi)$

KwTop  $\Delta\top$

KwEq  $\Delta\varphi \leftrightarrow \Delta\neg\varphi$

KtoMS  $\forall x\Delta\varphi \rightarrow \Delta(\varphi[y/x])$  (if  $\varphi[y/x]$  is admissible)

WB  $\forall x\Delta(\psi \rightarrow (\Delta\varphi \wedge \gamma)) \rightarrow \Delta(\psi \rightarrow (\forall x\Delta\varphi \wedge \gamma))$   
(if  $x$  occurs free only in  $\varphi$ )

## Rules:

MP  $\frac{\varphi, \varphi \rightarrow \psi}{\psi}$

REKw  $\frac{\vdash \varphi \leftrightarrow \psi}{\vdash \Delta\varphi \leftrightarrow \Delta\psi}$

RKwtoMA  $\frac{\vdash \varphi \rightarrow \Delta\psi}{\vdash \varphi \rightarrow \forall x\Delta\psi}$

The system is sound and complete over constant-domain models.

What we have covered:

- Brief history of Epistemic Logic.
- Important ideas and problems in the existing literature.
- Two dimensional EL and Dynamified EL.
- Beyond “knowing that”.

Looking forward to your presentations!