Epistemic Logic (X)

Logic of knowing value

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Background

A quote again...

"Classic" - a book which people praise and don't read.

– Mark Twain

We usually read the "digested" versions of the classics. However, they are not as "juicy" as the originals.

The classics (in logic) are usually rich in sincere considerations and questions behind the initiation of a new field, which should be appreciated beyond "archaeological" interests.

However, only the "most important" parts (sometimes "minor" things in the author's mind at the time of writing) survive after iterated filtering by influential authors and schools.

A classic paper in Dynamic Epistemic Logic

- Jan Plaza: Logics of public communications. In Proceedings of the 4th ISMIS Oak Ridge, TN: Oak Ridge National Laboratory, pp. 201-216. (1989) Unknown for a long time.
- Rediscovered in the late 90s after Gerbrandy and Groeneveld (1997) proposed a similar logic independently (in the Amsterdam tradition of update semantics).
- Reprinted in *Synthese* Volume 158, Issue 2, pp 165-179 (2007), with Hans van Ditmarsch's comments about the history of **DEL** before and after Plaza's paper, and content of the paper (pp 181-187).

What Plaza did

- Define the syntax and semantics of *public announcement logic* (PAL):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathcal{K}_i\varphi \mid \mathcal{K}w_i\varphi \mid \varphi + \varphi$$

$$\mathcal{M}, s \vDash \varphi + \psi \Leftrightarrow \mathcal{M}, s \vDash \varphi \text{ and } \mathcal{M}|_{\varphi}, s \vDash \psi$$

- $\varphi + \psi$ is essentially $\langle\varphi\rangle\psi$
- Discover the reduction to epistemic logic
- Give a complete proof system via reduction e.g., $\varphi + (\psi_1 \wedge \psi_2) \equiv (\varphi + \psi_1) \wedge (\varphi + \psi_2)$
- But that is only *half* of the paper!

A version of Plaza's system for PAL

Axiom Schemas	
TAUT	all the instances of tautologies
DISTK	$\mathcal{K}_i(\varphi \to \psi) \to (\mathcal{K}_i\varphi \to \mathcal{K}_i\psi)$
!ATOM	$[\psi]p \leftrightarrow (\psi \to p)$
!NEG	$[\psi]\neg\varphi \leftrightarrow (\psi \to \neg[\psi]\varphi)$
!CON	$[\psi](\varphi \land \chi) \leftrightarrow ([\psi]\varphi \land [\psi]\chi)$
!K	$[\psi]\mathcal{K}_i\varphi \leftrightarrow (\psi \to \mathcal{K}_i(\psi \to [\psi]\varphi))$
Rules	
NECK	$\dfrac{\varphi}{\mathcal{K}_i\varphi}$
RE	$\dfrac{\varphi \leftrightarrow \chi}{\psi \leftrightarrow \psi[\chi/\varphi]}$
MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$

One of the two running examples in Plaza's paper

Mr. Sum & Mr. Product

Mr. Puzzle: *I choose two natural numbers greater than 1 such that the sum is less than 100. I will tell the sum of the numbers only to Mr. Sum, and their product only to Mr. Product.*

He tells them.

Mr. Product: *I do not know the numbers.*
Mr. Sum: *I knew you didn't.*
Mr. Product: *But now I know!*
Mr. Sum: *So do I!*
What are the two numbers?

Knowing value

Know-value operator by Plaza (also Ma & Guo IJCAI83)

ELKv is defined as (where $c \in C$ is a constant symbol):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathcal{K}_i\varphi \mid \mathcal{K}v_i c$$

$\mathcal{K}v_i$ says "agent i knows [what] the value of c [is]"

ELKv is interpreted on FO-epistemic models with a *constant* domain $\mathcal{M} = \langle S, D, \{\sim_i \mid i \in \mathbf{I}\}, V, V_C\rangle$, where V_C assigns to each (non-rigid) $c \in C$ an $o \in D$ on each $s \in S$:

$$\mathcal{M}, s \vDash \mathcal{K}v_i c \iff \text{for any } t_1, t_2: \text{ if } s \sim_i t_1, s \sim_i t_2, \text{ then } V_C(c, t_1) = V_C(c, t_2).$$

Essentially the semantics says $\exists x \mathcal{K}(c \approx x)$. Knowing whether φ can be viewed as "knowing what the truth value of φ is".

Know-value operator by Plaza (also Ma, Guo IJCAI83)

ELKv can express "*i* knows that *j* knows the password but *i* doesn't know what exactly it is" by $\mathcal{K}_i\mathcal{K}v_j c \land \neg\mathcal{K}v_i c$.

The interaction between the two operators is crucial: it cannot be treated as $\mathcal{K}_i\mathcal{K}_j p \land \neg\mathcal{K}_i p$ which is inconsistent.

It is crucial in security protocol verification. Ways to capture "knowing what": e.g., introducing $has_i(m)$ as a basic proposition with a database of messages in the semantics.

See [Dechesne & Wang, Synthese 2010] for a survey on various knowledge in the security setting.

Know-value operator by Plaza (also Ma, Guo IJCAI83)

To handle the *Sum and Product* puzzle, Plaza extended **ELKv** with announcement operator (call it **PALKv**):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \land \varphi) \mid \mathcal{K}_i\varphi \mid \mathcal{K}v_i c \mid \langle\varphi\rangle\varphi$$

Plaza mentioned some axioms on top of $\mathbb{S}5$ and van Ditmarsch (2007) raised their completeness as an open problem.

Kv4	$\mathcal{K}v_i c$	$\to$	$\mathcal{K}_i\mathcal{K}v_i c$
Kv5	$\neg\mathcal{K}v_i c$	$\to$	$\mathcal{K}_i\neg\mathcal{K}v_i c$
KKv	$\langle\mathcal{K}_i\varphi\rangle\mathcal{K}v_i c$	$\leftrightarrow$	$\mathcal{K}_i\varphi \land \mathcal{K}v_i c$
	$\langle\mathcal{K}v_i c\rangle\mathcal{K}v_i d$	$\leftrightarrow$	$\mathcal{K}v_i c \land \mathcal{K}v_i d$
	$\langle\varphi\rangle\mathcal{K}v_i c$	$\to$	$\mathcal{K}_i(\varphi \to \langle\varphi\rangle\mathcal{K}v_i c)$
	$\langle\varphi\rangle\neg\mathcal{K}v_i c$	$\to$	$\mathcal{K}_i(\varphi \to \langle\varphi\rangle\neg\mathcal{K}v_i c)$

The last three can be derived from others.

$$
\begin{array}{llcl}
\mathsf{Kv4} & \mathcal{K}v_i c & \to & \mathcal{K}_i \mathcal{K}v_i c \\
\mathsf{Kv5} & \neg \mathcal{K}v_i c & \to & \mathcal{K}_i \neg \mathcal{K}v_i c \\
\mathsf{KKv} & \langle \mathcal{K}_i \varphi \rangle \mathcal{K}v_i c & \leftrightarrow & \mathcal{K}_i \varphi \land \mathcal{K}v_i c
\end{array}
$$

$$
\begin{array}{lll}
(i) & \mathcal{K}v_i c \leftrightarrow \mathcal{K}_i \mathcal{K}v_i c & \mathsf{Kv4, T} \\
(ii) & \langle \mathcal{K}v_i c \rangle \mathcal{K}v_i d \leftrightarrow \langle \mathcal{K}_i \mathcal{K}v_i c \rangle \mathcal{K}v_i d & \mathsf{RE} \\
(iii) & \langle \mathcal{K}_i \mathcal{K}v_i c \rangle \mathcal{K}v_i d \leftrightarrow \mathcal{K}_i \mathcal{K}v_i c \land \mathcal{K}v_i d & \mathsf{KKv} \\
(iv) & \langle \mathcal{K}v_i c \rangle \mathcal{K}v_i d \leftrightarrow (\mathcal{K}v_i c \land \mathcal{K}v_i d) & \mathsf{MP}(ii)\,(iii)
\end{array}
$$

$$
\begin{array}{lll}
(i) & \mathcal{K}v_i c \to \mathcal{K}_i \mathcal{K}v_i c & \mathsf{Kv4} \\
(ii) & \langle \varphi \rangle \mathcal{K}v_i c \to \langle \varphi \rangle \mathcal{K}_i Kv_i c & \mathsf{DISTA, NECA} \\
(iii) & \langle \varphi \rangle \mathcal{K}_i \mathcal{K}v_i c \leftrightarrow (\varphi \land \mathcal{K}_i(\varphi \to \langle \varphi \rangle \mathcal{K}v_i c)) & \mathsf{!K} \\
(iv) & \langle \varphi \rangle \mathcal{K}v_i c \to \mathcal{K}_i(\varphi \to \langle \varphi \rangle \mathcal{K}v_i c) & \mathsf{MP}(ii)(iii)
\end{array}
$$

Call $\mathbb{S}5$ plus Plaza's three axioms $\mathbb{PALKV}_p$.

Theorem (Wang & Fan IJCAI13)

$\theta = \langle p \rangle \mathcal{K}v_i c \wedge \langle q \rangle \mathcal{K}v_i c \rightarrow \langle p \vee q \rangle \mathcal{K}v_i c$ is not provable in $\mathbb{PALKV}_p$, thus $\mathbb{PALKV}_p$ is not complete.

Proof idea:

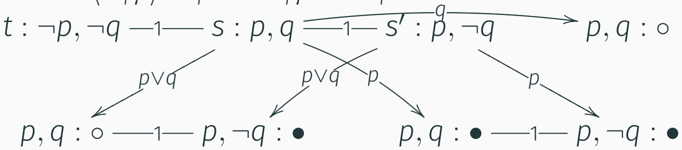
- define a class $\mathbb{C}$ of two-dimensional models (with $\xrightarrow{\varphi}$-labelled transitions) and a new semantics $\Vdash$ for **PALKv** such that:
- for all **PALKv** formulas φ: $\vdash \varphi \implies \mathbb{C} \Vdash \varphi$
- show that $\mathbb{C} \nVdash \theta$.

Cf. [Wang & Cao Synthese 2013] for the general method of constructing such semantics for **PAL** and incompleteness.

The idea of the counter model

Properties of $\mathbb{C}$ includes:

- Invariance for basic propositions, functionality of transitions, perfect recall, no miracles, replacement (about substitution)
- D-Invariance: if $s \stackrel{\mathcal{K}_i\varphi}{\to} t$, then $V_C(d, s) = V_C(d, t)$, in order to handle $\langle \mathcal{K}_i\varphi \rangle \mathcal{K}v_i c \leftrightarrow \mathcal{K}_i\varphi \wedge \mathcal{K}v_i c$.



$$s \nVdash \langle p \rangle \mathcal{K}v_i d \wedge \langle q \rangle \mathcal{K}v_i d \to \langle p \vee q \rangle \mathcal{K}v_i d$$

A bisimulation notion

A *d-bisimulation* between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a non-empty relation $Z \subseteq S_1 \times S_2$ such that if $s_1 Z s_2$ then the following requirements hold for all $i \in \mathbf{I}$ (besides the standard bis conditions):

Kv-Zig: if $t_1 \sim_i^1 s_1 \sim_i^1 t_1',$ and $V_C^1(c, t_1) \neq V_C^1(c, t_1')$ for some c then there exist $t_2, t_2' \in S_2$ such that $t_2 \sim_i^2 s_2 \sim_i^2 t_2',$ and $V_C^2(c, t_2) \neq V_C^2(c, t_2')$;

Kv-Zag: symmetric

We write $\mathcal{M}_1, s_1 \leftrightarrows_d \mathcal{M}_2, s_2$ iff there is a d-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ linking s_1 and s_2.

Proposition

If $\mathcal{M}_1, s_1 \leftrightarrows_d \mathcal{M}_2, s_2$, then $\mathcal{M}_1, s_1 \equiv_{\mathbf{ELKv}} \mathcal{M}_2, s_2$.

A reduction-based axiomatization is impossible

Now consider the following two epistemic models (using $\circ$ and $\bullet$ for the objects assigned to c):

$$s : p \circ \text{—1—} \neg p \circ \text{—1—} p \bullet \qquad s' : p \circ \text{—1—} \neg p \bullet$$

It is not hard to see that these two models are d-bisimiliar linking s and s'. However, we can distinguish s and s' easily by a **PALKv** formula $[p]\mathcal{K}v_1 c$.

Theorem (Wang & Fan IJCAI13)

PALKv is strictly more expressive than **ELKv**.

Conditionally knowing value

Conditionally knowing what

Axiomatizing **PALKv** is indeed hard (still open!). We propose a generalization of $\mathcal{K}v_i$ operator inspired by the relativized common knowledge operator (call it **ELKv**^{r}):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathcal{K}_i\varphi \mid \mathcal{K}v_i(\varphi, c)$$

where $\mathcal{K}v_i(\varphi, c)$ says "agent i knows what c is *given* φ", e.g., I know my password for this website given it is 4-digit.

$\mathcal{M}, s \vDash \mathcal{K}v_i(\varphi, c)$	$\Leftrightarrow$	for any $t_1, t_2 \in S$ such that $s \sim_i t_1$ and $s \sim_i t_2$: $\mathcal{M}, t_1 \vDash \varphi \& \mathcal{M}, t_2 \vDash \varphi$ implies $V_C(c, t_1) = V_C(c, t_2)$

Note that $\vDash \mathcal{K}v_i c \leftrightarrow \mathcal{K}v_i(\top, c)$.

Let **PALKv**^{r} be:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathcal{K}_i\varphi \mid \mathcal{K}v_i(\varphi, c) \mid \langle\varphi\rangle\varphi$$

PALKv^r looks more expressive than **PALKv** but in fact they are equally expressive.

Theorem (Wang & Fan IJCAI13)

The comparison of the expressive power of those logics are summarized in the following (transitive) diagram:

$$\begin{array}{ccc} ELKv^r & \longleftrightarrow & PALKv^r \\ \uparrow & & \updownarrow \\ ELKv & \longrightarrow & PALKv \end{array}$$

Translation t : **ELKv**^{r} $\to$ **PALKv**, g : **PALKv**^{r} $\to$ **ELKv**^{r}

$$\begin{array}{rcl} t(\mathcal{K}v_i(\varphi, d)) & = & \mathcal{K}_i\neg t(\varphi) \vee \hat{\mathcal{K}}_i\langle t(\varphi)\rangle\mathcal{K}v_i d \\ g(\langle\varphi\rangle\mathcal{K}v_i(\psi, d)) & = & g(\varphi) \wedge g(\mathcal{K}v_i(\langle\varphi\rangle\psi, d)) \end{array}$$

System $\mathbb{ELKVR}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\mathcal{K}_i(p \to q) \to (\mathcal{K}_i p \to \mathcal{K}_i q)$
T	$\mathcal{K}_i p \to p$
4	$\mathcal{K}_i p \to \mathcal{K}_i \mathcal{K}_i p$
5	$\neg \mathcal{K}_i p \to \mathcal{K}_i \neg \mathcal{K}_i p$
DISTKv^r	$\mathcal{K}_i(p \to q) \to (\mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p, c))$
Kv^{r}4	$\mathcal{K}v_i(p, c) \to \mathcal{K}_i \mathcal{K}v_i(p, c)$
Kv$^r\bot$	$\mathcal{K}v_i(\bot, c)$
Kv$^r\vee$	$\hat{\mathcal{K}}_i(p \wedge q) \wedge \mathcal{K}v_i(p, c) \wedge \mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p \vee q, c)$

Rules

MP	$\dfrac{p, p \to q}{q}$
NECK	$\dfrac{\varphi}{\mathcal{K}_i \varphi}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$

Kv$^r\vee$ was inspired by $\theta = \langle p \rangle \mathcal{K}v_i c \wedge \langle q \rangle \mathcal{K}v_i c \to \langle p \vee q \rangle \mathcal{K}v_i c$.

$\mathcal{K}_i \mathcal{K}v_i(\varphi, d) \leftrightarrow \mathcal{K}v_i(\varphi, d)$	T, Kv^{r}4
$\neg \mathcal{K}_i \mathcal{K}v_i(\varphi, d) \to \mathcal{K}_i \neg \mathcal{K}_i \mathcal{K}v_i(\varphi, d)$	5
$\neg \mathcal{K}v_i(\varphi, d) \to \mathcal{K}_i \neg \mathcal{K}v_i(\varphi, d)$	RE

Core ideas for the completeness

$\mathcal{K}v_i(\varphi, c)$ can be viewed as $\exists x \mathcal{K}_i(\varphi \to c = x)$ where x is a variable. Again, weak language vs. rich model.

Just using consistent sets as building blocks won't work:

$$p, d \mapsto \circ \text{------} p, d \mapsto \bullet$$

We need to saturate each maximal consistent set with:

- counterparts of atomic formulas such as $c \approx x$
- counterparts of $\mathcal{K}_i(\varphi \to c \approx x)$

We need to make sure the extra information is "consistent" by some conditions on the MCSs.

Definition (Wang & Fan AiML14)

Let MCS be the set of maximal consistent sets w.r.t. $\mathbb{ELKVR}$, and let $\mathbb{N}$ be the set of natural numbers. The canonical model $\mathcal{M}$ of $\mathbb{ELKVR}$ is a tuple $\langle S, \mathbb{N}, \{\sim_i \mid i \in \mathbf{I}\}, V, V_C\rangle$ where:

- S consists of all the triples $\langle \Gamma, f, g\rangle \in MCS \times \mathbb{N}^C \times (\mathbb{N} \cup \{\star\})^{\mathbf{I} \times \mathbf{ELKv}^r \times C}$ that satisfy the following three conditions:

  (i) $g(i, \psi, d) = \star$ iff $\mathcal{K}v_i(\psi, d) \wedge \hat{\mathcal{K}}_i\psi \notin \Gamma$,

  (ii) If $g(i, \varphi, d) \neq \star$ and $g(i, \psi, d) \neq \star$ then: $g(i, \varphi, d) = g(i, \psi, d)$ iff $\mathcal{K}v_i(\varphi \vee \psi, d) \in \Gamma$

  (iii) $\psi \wedge \mathcal{K}v_i(\psi, d) \in \Gamma$ implies $f(d) = g(i, \psi, d)$.

- $s \sim_i t$ iff $\{\varphi \mid \mathcal{K}_i\varphi \in s\} \subseteq t$ and $g_s(i) = g_t(i)$
- $V_C(d, s) = f_s(d)$

Lemma

Each maximal consistent set can be properly saturated with those counterparts.

Lemma

Each saturated MCS including $\hat{\mathcal{K}}\varphi$ has a saturated φ-successor.

Lemma

Each saturated MCS including $\neg Kv_i(\varphi, c)$ has **two** *saturated φ-successors which disagree about the value of c.*

A generalization of Axiom $\mathbf{Kv}^r\vee$ (U is a finite set of formulas) $\hat{\mathcal{K}}_i(\bigwedge U) \wedge \bigwedge_{\varphi\in U} \mathcal{K}v_i(\varphi, d) \to \mathcal{K}v_i(\bigvee U, d)$ is crucial.

The last lemma is broken down to two propositions:

Proposition

Given any $s \in S^c$ and any $i \in \mathbf{I}$, suppose there exist two (possibly identical) maximal consistent sets Γ_1 and Γ_2 such that:

(a) $\{\psi \mid \mathcal{K}_i\psi \in s\} \subseteq \Gamma_1 \cap \Gamma_2$

(b) *for any* $\mathcal{K}v_i(\theta, d) \in s$, $\theta \notin \Gamma_1 \cap \Gamma_2$.

then Γ_1 and Γ_2 can be extended into two states w, v in S^c such that $s \sim_i^c w$, $s \sim_i^c v$ and $f_w(d) \neq f_v(d)$.

Proposition

Given any $s \in S^c$ and any $i \in \mathbf{I}$, suppose $\neg\mathcal{K}v_i(\varphi, d) \in s$ then there are two (possibly identical) maximal consistent sets Γ_1 and Γ_2 such that:

- **(a')** $\{\varphi\} \cup \{\psi \mid \mathcal{K}_i\psi \in s\} \subseteq \Gamma_1 \cap \Gamma_2$
- **(b)** *for any $\mathcal{K}v_i(\theta, d) \in s$, $\theta \notin \Gamma_1 \cap \Gamma_2$.*

Let $Z = \{\psi \mid \mathcal{K}_i\psi \in s\} \cup \{\varphi\}$ and let $X = \{\neg\theta \mid \mathcal{K}v_i(\theta, d) \in s\}$. Note that due to $\mathsf{Kv}^r\bot$, X is non-empty. We want to build two consistent sets B and C such that $Z \subseteq B \cap C$ and $X \subseteq B \cup C$. Let $B_0 = Z \cup \{\neg\theta_0\}$ and let $C_0 = Z$ as the starting points. Then we build B_{n+1} and C_{n+1} based on the already defined B_n and C_n by adding $\neg\theta_{n+1}$ into one of them.

Theorem (Wang & Fan AiML14)

$\mathbb{ELKVR}$ *is sound and strongly complete for* ***ELKv***^{r}.

We can axiomatize multi-agent **PALKv**^{r} by adding the following reduction axiom schemas (call the resulting system $\mathbb{SPALKV}^r$):

!ATOM	$\langle\psi\rangle p \leftrightarrow (\psi \land p)$
!NEG	$\langle\psi\rangle\neg\varphi \leftrightarrow (\psi \land \neg\langle\psi\rangle\varphi)$
!CON	$\langle\psi\rangle(\varphi \land \chi) \leftrightarrow (\langle\psi\rangle\varphi \land \langle\psi\rangle\chi)$
!K	$\langle\psi\rangle\mathcal{K}_i\varphi \leftrightarrow (\psi \land \mathcal{K}_i(\psi \to \langle\psi\rangle\varphi))$
!Kv^r	$\langle\varphi\rangle\mathcal{K}v_i(\psi, c) \leftrightarrow (\varphi \land \mathcal{K}v_i(\langle\varphi\rangle\psi, c))$

Theorem (Xiong 14, Ding 14)

ELKv^r *(on epistemic models) is decidable.*

An alternative semantics

Axiomatizing ELKV^r over S5 frames [Wang and Fan AiML2014]

System $\mathbb{S}5$-$\mathbb{ELKVR}$

Axiom Schemas		Rules	
TAUT	all the instances of tautologies	MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
DISTK	$\mathcal{K}_i(p \to q) \to (\mathcal{K}_i p \to \mathcal{K}_i q)$	NECK	$\dfrac{\varphi}{\mathcal{K}_i \varphi}$
T	$\mathcal{K}_i p \to p$	SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$
4	$\mathcal{K}_i p \to \mathcal{K}_i \mathcal{K}_i p$	RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$
5	$\neg \mathcal{K}_i p \to \mathcal{K}_i \neg \mathcal{K}_i p$		
DISTKv^r	$\mathcal{K}_i(p \to q) \to (\mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p, c))$		
Kv^{r}4	$\mathcal{K}v_i(p, c) \to \mathcal{K}_i \mathcal{K}v_i(p, c)$		
Kv$^r\bot$	$\mathcal{K}v_i(\bot, c)$		
Kv$^r\vee$	$\hat{\mathcal{K}}_i(p \wedge q) \wedge \mathcal{K}v_i(p, c) \wedge \mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p \vee q, c)$		

Axiomatizing ELKV^r over arbitrary frames [Ding 2015]

System $\mathbb{ELKVR}$

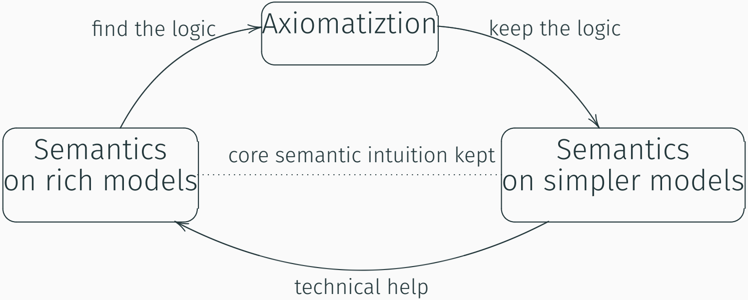
Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\mathcal{K}_i(p \to q) \to (\mathcal{K}_i p \to \mathcal{K}_i q)$
DISTKv^r	$\mathcal{K}_i(p \to q) \to (\mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p, c))$
Kv$^r\bot$	$\mathcal{K}v_i(\bot, c)$
Kv$^r\vee$	$\hat{\mathcal{K}}_i(p \wedge q) \wedge \mathcal{K}v_i(p, c) \wedge \mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p \vee q, c)$

- The completeness proofs are highly non-trivial due to the imbalance between the rich model and limited language.
- Suitable bisimulation notion for this logic was unknown.

Two questions and our key observation

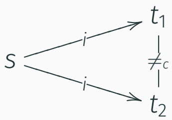
- How can it be connected to (normal) modal logic?
- How to rebalance the syntax and semantics?



Simplify the semantics while keeping the logic [Gu & Wang 16]

Observation: $\neg\mathcal{K}v_i(\varphi, c)$ can be viewed as a special diamond:

$$\mathcal{M}, s \vDash \neg\mathcal{K}v_i(\varphi, c) \quad\Leftrightarrow\quad \text{there exist } t_1, t_2 \in S \text{ such that } s \sim_i t_1 \text{ and } s \sim_i t_2 : \mathcal{M}, t_1 \vDash \varphi \text{ and } \mathcal{M}, t_2 \vDash \varphi \text{ but } V_C(c, t_1) \neq V_C(c, t_2)$$



We do not care about the exact values of c!

A modal language

To facilitate the comparison, we write $\neg\mathcal{K}v_i(\varphi, c)$ as $\Diamond_i^c\varphi$ and use the following language $\mathbf{MLKv}^r$:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_i\varphi \mid \Diamond_i^c\varphi$$

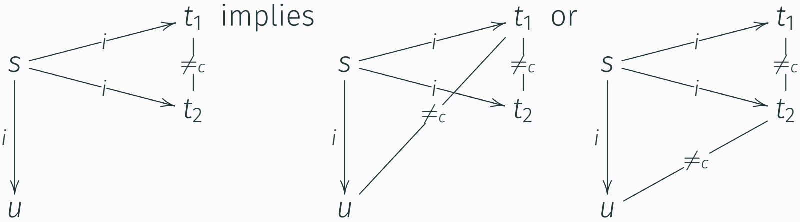
interpreted on Kripke models with binary and **ternary** relations $\langle S, \{\rightarrow_i : i \in I\}, \{R_i^c : i \in \mathbf{I}, c \in \mathbf{C}\}, V\rangle$, with extra conditions.

$$M, s \Vdash \Diamond_i^c\varphi \iff \exists u, v \text{: s.t. } sR_i^c uv \text{ and } \mathcal{M}, u \Vdash \varphi, \mathcal{M}, v \Vdash \varphi.$$

(1) $sR_i^c tu \iff sR_i^c ut$; (2) $sR_i^c uv$ only if $s \rightarrow_i u$ and $s \rightarrow_i v$;
(3) $sR_i^c tu$ and $s \rightarrow_i v$ implies that $sR_i^c tv$ or $sR_i^c uv$ holds;
(4) $sR_j^c tu$ for some $j \in \mathbf{I}$, $s \rightarrow_i t$ and $s \rightarrow_i u$ implies $sR_i^c tu$;
(5) $sR_j^c tu$ implies $t \neq u$.

An interesting property

$sR_i^c t_1 t_2$ and $s \to_i u$ implies that at least one of $sR_i^c t_1 u$ and $sR_i^c t_2 u$ holds



We show that (4)(5) do not matter: For any set $\Gamma \cup \{\varphi\}$ of **MLKv**^{r} formulas: $\Gamma \Vdash_{\mathbb{C}_{1-5}} \varphi \iff \Gamma \Vdash_{\mathbb{C}_{1-3}} \varphi \iff t(\Gamma) \vDash t(\varphi)$ where t translates **MLKv**^{r} formulas back to **ELKv**^{r}.

Recall the system for ELKV^r.

System $\mathbb{ELKVR}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\mathcal{K}_i(p \to q) \to (\mathcal{K}_i p \to \mathcal{K}_i q)$
DISTKv^r	$\mathcal{K}_i(p \to q) \to (\mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p, c))$
Kv$^r\bot$	$\mathcal{K}v_i(\bot, c)$
Kv$^r\vee$	$\hat{\mathcal{K}}_i(p \wedge q) \wedge \mathcal{K}v_i(p, c) \wedge \mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p \vee q, c)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\mathcal{K}_i \varphi}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$

In the new language:

- DISTKv^r: $\Box_i(p \to q) \to (\Box_i^c \neg q \to \Box_i^c \neg p)$ equivalent to $\Box_i(p \to q) \to (\Box_i^c p \to \Box_i^c q)$ under SUB and RE.
- Kv$^r\vee$: $\Diamond_i(p \wedge q) \wedge \Diamond_i^c(p \vee q) \to (\Diamond_i^c p \vee \Diamond_i^c q)$
- Kv$^r\bot$: $\Box_i^c \top$

A new look at the axiomatization

System $\mathbb{SMLKVR}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
DISTKv^r	$\Box_i(p \to q) \to (\Box_i^c p \to \Box_i^c q)$
Kv$^r\vee$	$\Diamond_i(p \wedge q) \wedge \Diamond_i^c(p \vee q) \to (\Diamond_i^c p \vee \Diamond_i^c q)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i \varphi}$
NECK^r	$\dfrac{\varphi}{\Box_i^c \varphi}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$

We replace $\Box_i^c \top$ by a necessitation rule NECK^r.

Theorem (Gu & Wang AiML16)

$\mathbb{SMLKVR}$ *is sound and complete w.r.t.* $\mathbb{C}_{1-3}$ *(and* $\mathbb{C}_{1-5}$*).*

A relatively easy canonical model construction suffices (3 page).

A New look at the axiomatization

System $\mathbb{SMLKVR}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
DISTKv^r	$\Box_i(p \to q) \to (\Box_i^c p \to \Box_i^c q)$
Kv$^r\vee$	$\Diamond_i(p \wedge q) \wedge \Diamond_i^c(p \vee q) \to (\Diamond_i^c p \vee \Diamond_i^c q)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i \varphi}$
NECK^r	$\dfrac{\varphi}{\Box_i^c \varphi}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$

Note that $\Diamond_i^c(\varphi \vee \psi) \to (\Diamond_i^c \varphi \vee \Diamond_i^c \psi)$ does not hold. Moreover, $\Box_i^c(\varphi \to \psi) \to (\Box_i^c \varphi \to \Box_i^c \psi)$ does not hold neither, thus the logic is **not** normal.

However, this is only the appearance.

Disguised normal modal logic

$\Diamond_i^c$ is essentially a **binary** diamond!

In `MLKvr` we only allow $\Diamond_i^c(\varphi, \varphi)$. Let `MLKvb` be the language with $\Diamond_i^c(\varphi, \psi)$.

$\Diamond_i^c(\varphi, \psi)$ has the standard semantics for (polyadic) normal modal logic:

$$M, s \Vdash \Diamond_i^c(\varphi, \psi) \iff \exists\, u, v\text{: s.t. } sR_i^c uv \text{ and } \mathcal{M}, u \Vdash \varphi,\ \mathcal{M}, v \Vdash \psi.$$

The generalization does not increase expressivity

Proposition

MLKvb *is equally expressive as* MLKvr *over* $\mathbb{C}_{1-3}$.

$\Diamond_i^c(\varphi, \psi)$ is equivalent to the disjunction of the following:

- $\Diamond_i^c \varphi \wedge \Diamond_i \psi$
- $\Diamond_i^c \psi \wedge \Diamond_i \varphi$
- $\Diamond_i \varphi \wedge \Diamond_i \psi \wedge \neg \Diamond_i^c \varphi \wedge \neg \Diamond_i^c \psi \wedge \Diamond_i^c (\varphi \vee \psi)$

A normal polyadic modal logic

System $\mathbb{SMLKVB}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
DISTBK	$\Box_i^c(p \to q, r) \to (\Box_i^c(p, r) \to \Box_i^c(q, r))$
SYM	$\Box_i^c(p, q) \to \Box_i^c(q, p)$
INCL	$\Diamond_i^c(p, q) \to \Diamond_i p$
DISBK	$\Diamond_i^c(p, q) \wedge \Diamond_i r \to \Diamond_i^c(p, r) \vee \Diamond_i^c(q, r)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i \varphi}$
NECKvb	$\dfrac{\varphi}{\Box_i^c(\varphi, \psi)}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$

Theorem (Gu & Wang AiML16)

$\mathbb{SMLKVB}$ is sound and complete w.r.t. $\mathbb{C}_{1-3}$ and $\mathbb{C}_{1-5}$.

$\mathbb{SMLKVB}$ can drive all the axioms in $\mathbb{SMLKVR}$.

The completeness proof is now mostly routine (one page)

$\mathcal{M}^c = \langle S, \{\rightarrow_i : i \in \mathbf{I}\}, \{R_i^c : i \in \mathbf{I}, c \in \mathbb{C}\}, V\rangle$

- S is the set of all maximal $\mathbb{SMLKVB}$-consistent sets of MLKvb formulas,
- $s \rightarrow_i t \iff \{\varphi : \Box_i\varphi \in s\} \subseteq t$,
- $sR_i^c tu \iff (1)\ \{\varphi : \Box_i\varphi \in s\} \subseteq t \cap u$ and (2) for any $\Box_i^c(\varphi, \psi) \in s$, $\varphi \in t$ or $\psi \in u$.
- $V(s) = \{p : p \in s\}$.

SYM, INCL, and DISBK are canonical for the corresponding properties 1-3.

ELKvr as a normal modal logic

ELKv^r can be viewed as a disguised normal modal logic!

Standard techniques apply:

- Canonical model for free.
- Bisimulation for free.
- ? Decision procedure

These will help us in solving problems about the original **ELKv**^{r}.

Definition (Bisimulation)

Let $\mathcal{M}_1 = \langle S_1, \{\to_i^1 : i \in I\}, \{R_i^c : i \in I, c \in \mathbb{C}\}, V_1\rangle$, $\mathcal{M}_2 = \langle S_2, \{\to_i^2 : i \in I, c \in \mathbb{C}\}, \{Q_i^c : i \in I\}, V_2\rangle$ be two models for **MLKvb** (also for **MLKv**^{r}). A $\mathbb{C}$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a non-empty binary relation $Z \subseteq S_1 \times S_2$ such that for all $s_1 Z s_2$, the following conditions are satisfied:

- **Inv** : $V_1(s_1) = V_2(s_2)$;
- **Zig** : $s_1 \to_i^1 t_1 \Rightarrow \exists t_2$ such that $s_2 \to_i^2 t_2$ and $t_1 Z t_2$;
- **Zag** : $s_2 \to_i^2 t_2 \Rightarrow \exists t_1$ such that $s_1 \to_i^1 t_1$ and $t_1 Z t_2$;
- **Kvb-Zig** : $s_1 R_i^c t_1 u_1 \Rightarrow \exists t_2, u_2 \in S_2$ such that $t_1 Z t_2$, $u_1 Z u_2$ and $s_2 Q_i^c t_2 u_2$;
- **Kvb-Zag** : $s_2 Q_i^c t_2 u_2 \Rightarrow \exists t_1, u_1 \in S_1$ such that $t_1 Z t_2$, $u_1 Z u_2$ and $s_1 R_i^c t_1 u_1$.

A simpler logic

Plaza's unconditional language:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathcal{K}_i\varphi \mid \mathcal{K}v_i c$$

is essentially:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_i\varphi \mid \Box_i^c\bot$$

System $\mathbb{SMLKV}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
INCLT	$\Diamond_i^c\top \to \Diamond_i\top$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i\varphi}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$

Further directions

Adding dynamics

Enrich **ELKv**^{r} with *public inspection* $[c]$:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathcal{K}_i\varphi \mid \mathcal{K}v_i(\varphi, c) \mid [c]\varphi$$

$$\boxed{\mathcal{M}, s \vDash [c]\varphi \quad \Leftrightarrow \quad \mathcal{M}|_c^s, s \vDash \varphi}$$

where $\mathcal{M}|_c^s$ is defined as the tuple $\langle S', D, \sim|_{S'\times S'}, V|_{\mathbf{C}\times S'}, V_{\mathbf{C}}|_{S'}\rangle$ where $S' = \{s' \mid V_{\mathbf{C}}(c, s') = V_{\mathbf{C}}(c, s)\}$. Note that the relativization here is *not global*.

PSELKv^r is more expressive than **ELKv**^{r}: $[c]$ cannot be reduced.

How to axiomatize it?

It is hard and still open, but you can:

- Restrict the language: to only allow $\mathcal{K}v_i c$ and $[c]\varphi$ [van Eijck, Gattinger, Wang ICLA17]
- Enrich the language: with equalities, and conditional operator for all the combinations of values and propositions [Baltag AiML16]

Connections with automata theory...

Connections to dependence logic and inquisitive logic

We can define the knowledge of dependency as following:

$$Kd_i(c, d) := \mathcal{K}_i[c]\mathcal{K}v_i d$$

$$\mathcal{M}, s \vDash Kd_i(c, d) \Leftrightarrow \text{for all } t_1 \sim_i s, t_2 \sim_i s : V_{\mathbf{C}}(c, t_1) = V_{\mathbf{C}}(c, t_2) \implies V_{\mathbf{C}}(d, t_1) = V_{\mathbf{C}}(d, t_2)$$

I know the value of d depends on the value of c, it can be expressed by $=(c, d)$ in *dependence logic* by Väänänen, taking the epistemic possibilities as "teams" in their setting.

It is also the knowledge of the entailment of interrogatives in first-order inquisitive logic:

$$K_i(?\text{value } c \to ?\text{value } d) := Kd_i(c, d)$$

Weakly Aggregative Logic

Modal logic with diagnol modalities as special cases on polyadic modal logic:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

It can be defined on models with n-ary relations.

$\mathcal{M}, w \models \Box\varphi$	iff	for all $v_1, \dots v_n \in W$ with $Rwv_1 \dots, v_n, \mathcal{M}, v_i \models \varphi$ for some $i \leq n$.
$\mathcal{M}, w \models \Diamond\varphi$	iff	there are $v_1, \dots v_n \in W$ st. $Rwv_1 \dots, v_n$ and $\mathcal{M}, v_i \models \varphi$ for all $i \leq n$.

The following is valid over models with $n + 1$-ary relations:

$$\Box p_0 \wedge \cdots \wedge \Box p_n \to \Box \bigvee_{(0 \leq i < j \leq n)} (p_i \wedge p_j).$$

It does not have Craig interpolation! [Liu, Ding and Wang 2019]

Take-home messages

- Do (re)read the classics
- To handle the weak language in the completeness proof, we need to provide extra information in building the canonical model. The maximal consistent sets do not suffice.
- Conditional knowing value logic is a disguised normal modal logic with binary and ternary modalities.