Epistemic Logic VI

The Dynamic Turn (B)

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Two basic questions

Axiomatizations via reduction

A new axiomatization

Recap: Public Announcement Logic (PAL)

The language of *Public Announcement Logic* (PAL):

$$\phi \quad ::= \quad \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_i\phi \mid [\phi]\phi \text{ (also write } [!\phi]\phi)$$

Interpreted on (usually S5) Kripke models $\mathcal{M} = (S, \{\rightarrow_i\}_{i\in I}, V)$:

$$\begin{array}{|rcl|} \hline \mathcal{M}, s \vDash \Box_i\psi & \Leftrightarrow & \forall t : s \rightarrow_i t \implies \mathcal{M}, t \vDash \psi \\ \mathcal{M}, s \vDash [\psi]\phi & \Leftrightarrow & \mathcal{M}, s \vDash \psi \text{ implies } \mathcal{M}|_\psi, s \vDash \phi \\ \hline \end{array}$$

where $\mathcal{M}|_\psi = (S', \{\rightarrow'_i \mid i \in \mathrm{I}\}, V')$ such that: $S' = \{s \mid \mathcal{M}, s \vDash \psi\}$, $\rightarrow'_i = \rightarrow_i |_{S' \times S'}$ and $V'(p) = V(p) \cap S'$.

$$\mathcal{M}, s_1 \vDash \neg\Box_1 p \wedge [p]\Box_1 p$$

Recap: PA + your choice

Axiom Schemas	
TAUT	all the instances of tautologies
DISTK	$\Box_i(\phi \to \psi) \to (\Box_i\phi \to \Box_i\psi)$
!ATOM	$[\psi]p \leftrightarrow (\psi \to p)$
!NEG	$[\psi]\neg\phi \leftrightarrow (\psi \to \neg[\psi]\phi)$
!CON	$[\psi](\phi \wedge \chi) \leftrightarrow ([\psi]\phi \wedge [\psi]\chi)$
!K	$[\psi]\Box_i\phi \leftrightarrow (\psi \to \Box_i[\psi]\phi)$
Rules	
NECK	$\dfrac{\phi}{\Box_i\phi}$
MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
Your choice	
RE	$\dfrac{\phi \leftrightarrow \chi}{\psi \leftrightarrow \psi[\chi/\phi]}$
!COM	$[\psi][\chi]\phi \leftrightarrow [\psi \wedge [\psi]\chi]\phi$

Plaza's notation may help to see reduction axioms

In Plaza's paper: $\phi + \psi := \langle\phi\rangle\psi$, $\equiv := \leftrightarrow$, under the rule of RE.

$$\phi + p \equiv \phi \wedge p$$

$$\phi + \top \equiv \phi$$

$$\phi + \bot \equiv \bot$$

$$\phi + (\psi_1 \wedge \psi_2) \equiv (\phi + \psi_1) \wedge (\phi + \psi_2)$$

$$\phi + (\psi_1 \vee \psi_2) \equiv (\phi + \psi_1) \vee (\phi + \psi_2)$$

$$\phi + \neg\psi \equiv \phi \wedge \neg(\phi + \psi)$$

$$\phi + (\psi_1 \to \psi_2) \equiv \phi \wedge (\phi + \psi_1 \to \phi + \psi_2)$$

$$\phi + (\psi_1 \equiv \psi_2) \equiv \phi \wedge (\phi + \psi_1 \equiv \phi + \psi_2)$$

$$\phi + \Box_i\psi \equiv \phi \wedge \Box_i(\phi \to \phi + \psi)$$

Interpretation of PA^+ in system K:

For all $\phi \in \mathsf{PAL}$: $\vdash_{\mathsf{PA}^+} \phi \iff \vdash_{\mathsf{K}} t(\phi)$ and $\vdash_{\mathsf{PA}^+} \phi \leftrightarrow t(\phi)$.

Plaza's notation may help to see reduction axioms

$\phi + \psi \not\equiv \psi + \phi, \phi + \phi \not\equiv \phi$ but...

The following are provable theorems:

$$\top + \phi \equiv \phi$$
$$\bot + \phi \equiv \bot$$
$$\phi + (\psi + \chi) \equiv (\phi + \psi) + \chi$$
$$\phi + \psi \rightarrow \phi$$
$$(\phi_1 + \cdots + \phi_i + \cdots + \phi_n) \rightarrow (\phi_1 + \cdots + \phi_i)$$
$$(\phi + \psi_1) \wedge (\phi + (\psi_1 \rightarrow \psi_2)) \rightarrow \phi + \psi_2$$

There are also algebraic semantics for PAL.

The reduction technique turns out to be extremely useful in many applications and thus dominates the field of DEL.

- Logic is more than it appears!
- Update-closeness may be considered as a desired property of a logic: it shows the logic has enough pre-encoding power [van Benthem et al., 2006].
- Regressive analysis of the knowledge updates.
- Analogy of difference equations in dynamical system.
- Also good for lazy guys to have some “results”.
- The orthodox programme of DEL:
  static logic+dynamic operators+reduction

There can be lots of variations

Summary of Part (II) of van Benthem's *Logicial Dynamics* book

chapter	representation	transformation
PAL	epistemic model (EM)	relativization
DEL	EM	product update
awareness	EM + accessible sets	relativization and realization
issue management	EM + issue relations	link-intersection and product updat
belief	EM + plausibility relations	lexicographic/conservative upgrade
probability	EM + probability distributions	probabilistic product update
preference	modal betterness model	like Ch.7, defined by PDL programs
games	EM + moves (extensive games)	relativization and product update
procedures	EM + protocols	relativization and product update
groups	doxastic model	priority update

Two basic questions

The first question

In some published papers, PA and its variants are mentioned as complete systems. Is PA really complete?

Unfortunately, PA and many of its siblings are not complete, and in some cases the flaws cannot be fixed.

The second question

Can we give meaningful axiomatizations without those reduction axioms and the reduction proof method?

Yes, we can! *Not all the axiomatizations are born equal!*

We will give a general axiomatization method inspired by Epistemic Temporal Logic. It will tell us what exactly is assumed in Dynamic Epistemic Logic.

We will look at the answers in detail [Wang and Cao, 2013].

Axiomatizations via reduction

First question

Is PA complete?

$$\vDash \phi \implies \vDash t(\phi) \overset{\text{completeness of K}}{\implies} \vdash_{\mathsf{K}} t(\phi) \implies \vdash_{\mathsf{PA}} t(\phi) \overset{\textit{Rd.Axioms}}{\implies} \vdash_{\mathsf{PA}} \phi$$

$$t([\psi][\chi]\phi) = t([\psi]t([\chi]\phi))$$

$$t'([\psi][\chi]\phi) = t'([\psi \wedge [\psi]\chi]\phi)$$

The first translation needs RE, the second translation needs !COM in the proof system.

Negative Answer

PA is not complete!

We need to show that there exists ϕ: $\vDash \phi$ but $\nvdash_{\mathsf{PA}} \phi$.

A general proof strategy to show some formula is not derivable in a proof system S: design a non-standard semantics $\Vdash$ which validates the axioms and rules in S.

Thus for all $\phi : \vdash_{\mathsf{S}} \phi \implies \Vdash \phi$. Then from $\nVdash \phi$ we have $\nvdash_{\mathsf{S}} \phi$.

A non-standard semantics

Goal: design a semantics to validate PA but not !COM (nor RE).
Given a Kripke model over $\mathcal{M} = (S, \{\to_i \mid i \in I\}, V)$, the truth value of a PAL formula ϕ at a state s in $\mathcal{M}$ is recursively defined as based on $\Vdash_\rho$ where ρ is a formula in the language of PAL:

$$
\begin{array}{rcl}
\mathcal{M}, s \Vdash \phi & \Leftrightarrow & \mathcal{M}, s \Vdash_\top \phi \\
\mathcal{M}, s \Vdash_\rho \top & \Leftrightarrow & \text{always} \\
\mathcal{M}, s \Vdash_\rho p & \Leftrightarrow & p \in V(s) \\
\mathcal{M}, s \Vdash_\rho \neg\phi & \Leftrightarrow & \mathcal{M}, s \nVdash_\rho \phi \\
\mathcal{M}, s \Vdash_\rho \phi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \Vdash_\rho \phi \text{ and } \mathcal{M}, s \Vdash_\rho \psi \\
\mathcal{M}, s \Vdash_\rho [\psi]\phi & \Leftrightarrow & \mathcal{M}, s \Vdash_\top \psi \text{ implies } \mathcal{M}, s \Vdash_{\rho\wedge\psi} \phi \\
\mathcal{M}, s \Vdash_\rho \Box_i\phi & \Leftrightarrow & \forall t \text{ such that } sR_it : \mathcal{M}, t \Vdash_\top \rho \text{ implies } \mathcal{M}, t \Vdash_\rho \phi
\end{array}
$$

We say ϕ is *valid* w.r.t. $\Vdash$ if $\Vdash \phi$ (equivalently $\Vdash_\top \phi$).

It is handy to show $\Vdash \rho \leftrightarrow \rho'$ then $\mathcal{M}, s \Vdash_\rho \phi \Leftrightarrow \mathcal{M}, s \Vdash_{\rho'} \phi$ for any ϕ and any $\mathcal{M}, s$.

Consider the following (S5) model $\mathcal{M}$ with two worlds s, v:

$\mathcal{M}, s \Vdash \neg\Box_i p \iff \mathcal{M}, s \nVdash_\top \Box_i p \iff (\exists t \rhd_i s : \mathcal{M}, t \Vdash_\top \top \text{ and } \mathcal{M}, t \nVdash_\top p)$. Since $p \notin V(v)$ and $s \xrightarrow{i} v$, $\mathcal{M}, s \Vdash \neg\Box_i p$.

$\mathcal{M}, s \Vdash_p \Box_i p \iff (\forall t \rhd_i s : \mathcal{M}, t \Vdash_\top p \text{ implies } \mathcal{M}, t \Vdash_p p)$.

Clearly, $\mathcal{M}, s \Vdash_p \Box_i p$. Similarly $\mathcal{M}, s \Vdash_{\top\wedge p} \Box_i p$.

$\mathcal{M}, s \Vdash [p][\neg\Box_i p]\bot \iff (\mathcal{M}, s \Vdash_\top p \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge p} [\neg\Box_i p]\bot) \iff (\mathcal{M}, s \Vdash_\top \neg\Box_i p \text{ implies } \mathcal{M}, s \Vdash_{\top\wedge p\wedge\neg\Box_i p} \bot)$. Thus $\mathcal{M}, s \nVdash [p][\neg\Box_i p]\bot$.

On the other hand, it is easy to verify that
$\mathcal{M}, s \Vdash [p \wedge [p]\neg\Box_i p]\bot$.

Recall the other axioms

Axiom Schemas	
DIST!	$[\psi](\phi \to \chi) \to ([\psi]\phi \to [\psi]\chi)$
!COM	$[\psi][\chi]\phi \leftrightarrow [\psi \wedge [\psi]\chi]\phi$
Rules	
NEC!	$\dfrac{\phi}{[\psi]\phi}$
RE	$\dfrac{\phi \leftrightarrow \chi}{\psi \leftrightarrow \psi[\chi/\phi]}$

$$\texttt{!COM} : [\psi][\chi]\phi \leftrightarrow [\psi \land [\psi]\chi]\phi \quad ✗$$

$$\texttt{!COM}\land : [\psi][\chi]\phi \leftrightarrow [\psi \land \chi]\phi$$

Theorem
For all PAL *formulas* ϕ*:* $\vdash_{PA+\texttt{DIST!}} \phi$ *implies* $\Vdash \phi$.

Lemma
None of $\texttt{!COM}$, $\texttt{NEC!}$, $\texttt{RE!}$, $\texttt{RE}$ *is valid under* $\Vdash$.

Theorem
$PA + \texttt{DIST!}$ *is not complete.*

Theorem
$PA + \texttt{!COM}\land$ *is sound and complete w.r.t.* $\Vdash$.

To complete the whole picture

We have seen that $\mathsf{PA} + \texttt{DIST!}$ is not complete but $\mathsf{PA} + \texttt{NEC!} + \texttt{DIST!}$ is complete (why?). So, what about $\mathsf{PA} + \texttt{NEC!}$? We need to design a new semantics.

Theorem
$\texttt{DIST!}$ *is not derivable from* $\mathit{PA} + \texttt{NEC!}$.

As an immediate corollary:

Corollary
$\mathit{PA} + \texttt{NEC!}$ *is not complete w.r.t. standard semantics* $\vDash$.

Conclusion of the answer to question 1

Summary of the results (PA can be replaced by PAS5, see [Wang and Cao, 2013]):

derivable/admissible in PA	not derivable/admissible in PA
WDIST!, FUNC, RE$\neg$, RE$\wedge$, RE$\Box$	!COM, DIST!, SDIST!, PRE, !K', NEC!, RE!, RE
sound & complete systems	sound & incomplete systems
PA-!CON+DIST!+NEC!, PA+PRE+NEC! PA+RE, PA+!COM	PA+!K'+PRE+DIST!+!RE, PA+NEC!

The lesson that we learned:

There may be different ways to conduct the reductions in DEL logics which require different facilities in the proof system.

Make your choice carefully! Constructing alternative semantics can help us to understand the merit of the original semantics

An alternative context-dependent semantics

The context-dependent semantics gives us a lot more freedom in designing the semantics for dynamic epistemic logic. The updates will only change the *context*, not the model:

$$
\begin{array}{rcl}
\mathcal{M}, s \Vdash \phi & \Leftrightarrow & \mathcal{M}, s \Vdash_\top \phi \\
\mathcal{M}, s \Vdash_\chi \top & \Leftrightarrow & \text{always} \\
\mathcal{M}, s \Vdash_\chi p & \Leftrightarrow & p \in V(s) \\
\mathcal{M}, s \Vdash_\chi \neg\phi & \Leftrightarrow & \mathcal{M}, s \nVdash_\chi \phi \\
\mathcal{M}, s \Vdash_\chi \phi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \Vdash_\chi \phi \text{ and } \mathcal{M}, s \Vdash_\chi \psi \\
\mathcal{M}, s \Vdash_\chi \Box_i \psi & \Leftrightarrow & \forall t : (s \rightarrow_i t \text{ and } \mathcal{M}, t \Vdash_\top \chi) \text{ implies } \mathcal{M}, t \Vdash_\chi \psi \\
\mathcal{M}, s \Vdash_\chi [\psi]\phi & \Leftrightarrow & \mathcal{M}, s \Vdash_\chi \psi \text{ implies } \mathcal{M}, s \Vdash_{\chi \wedge [\chi]\psi} \phi
\end{array}
$$

We can show: $\mathcal{M}, s \vDash \phi \iff \mathcal{M}, s \Vdash \phi$.

Similar semantics leads to a Gentzen-style sequent system for PAL (Maffezioli and Negri 11).

A new axiomatization

Reduction axioms: must-do or coincidence?

Can we give meaningful axiomatizations without those reduction axioms and the reduction proof method?

Yes, we can!

We will give a new axiomatization with a general proof method inspired by Epistemic Temporal Logic.

Let us go back to the standard method in normal modal logic.

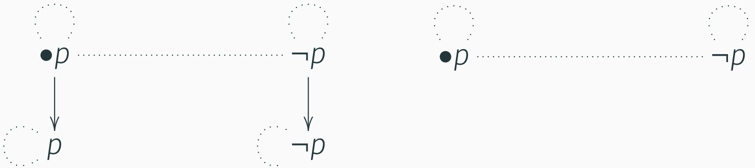
Background: ETL and DEL

They are semantics-driven two-dimensional modal logics:

	language	model	semantics
ETL	time+K	temporal+epistemic	Kripke-like
DEL	K+events	epistemic	Kripke+*dynamic*

$\neg Kp \wedge F\, Kp$

$\neg Kp \wedge [!p]Kp$



Background

	language	model	semantics
ETL	time+K	temporal+epistemic	Kripke-like
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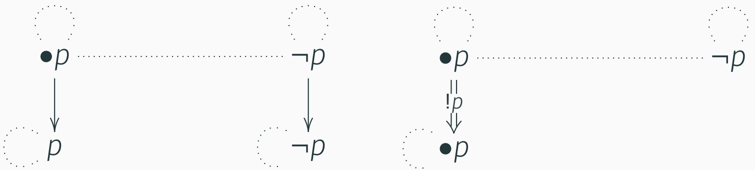
$\neg Kp \wedge F\, Kp$ $\qquad\qquad$ $\neg Kp \wedge [!p]Kp$



Dynamic semantics: the **meaning** of an event is the **change** it brings to the knowledge states.

Bridging the two

An earlier observation: Iterated updating epistemic structures generates special ETL-style "super models" [van Benthem et al., 2009].

Our approach: relate DEL and ETL via axioms.

New method

Basic idea: treat $[\psi]$ as a **normal** modality interpreted on the standard two-dimensional ETL models with labelled transitions:

$$(S, \to, \{\xrightarrow{\psi} \mid \psi \in \mathsf{PAL}\}, V)$$

We call $(S, \to, V)$ the *Epistemic core* of $\mathcal{M}$ (notation $\mathcal{M}^-$).

$$\boxed{\mathcal{M}, s \Vdash [\psi]\phi \quad \Leftrightarrow \quad \forall t : s \xrightarrow{\psi} t \text{ implies } \mathcal{M}, t \Vdash \phi}$$

Proof strategy: find a class of ETL-style models $\mathbb{C}$ and show the following:

$$\vDash \phi \implies \mathbb{C} \Vdash \phi \implies \vdash_{\mathsf{S}} \phi.$$

for some system S. $\implies$ can be strengthened to $\iff$.

Definition (Normal ETL models w.r.t. PAL)

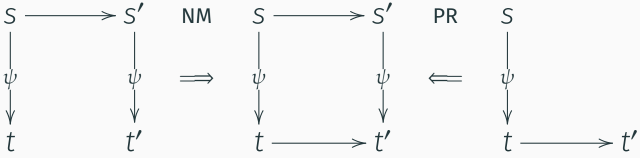
An ETL-like model $\mathcal{M} = (S, \to, \{\xrightarrow{\psi} \mid \psi \in \mathsf{PAL}\}, V)$ for PAL is called *normal* if the following properties hold for any s, t in $\mathcal{M}$:

U-Executability For any PAL formula ψ: $\mathcal{M}, s \Vdash \psi$ iff s has outgoing ψ-transitions.

U-Invariance if $s \xrightarrow{\psi} t$ then $s \in V(p) \iff t \in V(p)$ for all $p \in \mathbf{P}$.

U-Zig (NM) if $s \to s'$, $s' \xrightarrow{\psi} t'$ and $s \xrightarrow{\psi} t$ then $t \to t'$.

U-Zag (PR) if $s \xrightarrow{\psi} t$ and $t \to t'$ then there exists an s' such that $s \to s'$ and $s' \xrightarrow{\psi} t'$.



These are the properties of (synchronous) no miracles and perfect recall.

Same language, two logics: $\langle \mathsf{PAL}, \mathbb{M}, \vDash \rangle$ and $\langle \mathsf{PAL}, \mathbb{C}, \Vdash \rangle$. We want to show that $\mathbb{M} \vDash \phi \implies \mathbb{C} \Vdash \phi$.

We can show for any ϕ any normal model $\mathcal{N}$:

$$\mathcal{N}, s \Vdash \phi \iff \mathcal{N}^-, s \vDash \phi$$

An inductive proof suffices where the following step is crucial:

$$\mathcal{N}, s \Vdash [\psi]\chi \iff \mathcal{N}^-, s \vDash [\psi]\chi$$

To show this we prove the following by

$$\text{If } s \xrightarrow{\psi} t \text{ then } \mathcal{N}^-, t \,\underline{\leftrightarrow}\, \mathcal{N}^-|_\psi, s$$

$\underline{\leftrightarrow}$ is the bisimilarity relation w.r.t. the epistemic relation. Why it is enough?

Definition (Bisimulation w.r.t. $\to$)

A binary relation Z is called a *bisimulation* between two Kripke models $\mathcal{M}$ and $\mathcal{N}$, if sZt and whenever wZv the following hold:

Invariance $p \in V^{\mathcal{M}}(w)$ iff $p \in V^{\mathcal{N}}(v)$,

Zig if $w \to w'$ for some w' in $\mathcal{M}$ then there is a $v' \in S^{\mathcal{N}}$ with $v \to v'$ and $w'Zv'$,

Zag if $v \to v'$ for some v' in $\mathcal{N}$ then there is a $w' \in S^{\mathcal{M}}$ with $w \to w'$ and $w'Zv'$.

If there exists a bisimulation between $\mathcal{M}$, and $\mathcal{N}$ linking s in $\mathcal{M}$ to t in $\mathcal{N}$ then we say pointed models $\mathcal{M}, s$ and $\mathcal{N}, t$ are bisimilar (notation: $\mathcal{M}, s \leftrightarroweq \mathcal{N}, t$).

PAL formulas are invariant under bisimilarity: if $\mathcal{M}, s \leftrightarroweq \mathcal{N}, t$ then for all PAL formula ϕ: $\mathcal{M}, s \vDash \phi \iff \mathcal{N}, t \vDash \phi$.

System PAN

Axiom Schemas		Rules	
TAUT	all the instances of tautologies	MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
DISTK	$\Box(\phi \to \chi) \to (\Box\phi \to \Box\chi)$	NECK	$\dfrac{\phi}{\Box\phi}$
DIST!	$[\psi](\phi \to \chi) \to ([\psi]\phi \to [\psi]\chi)$	NEC!	$\dfrac{\phi}{[\psi]\phi}$
INV	$(p \to [\psi]p) \land (\neg p \to [\psi]\neg p)$		
EXE	$\langle\psi\rangle\top \leftrightarrow \psi$		
NM	$\Diamond\langle\psi\rangle\phi \to [\psi]\Diamond\phi$		
PR	$\langle\psi\rangle\Diamond\phi \to \Diamond\langle\psi\rangle\phi$		

where $p \in \mathbf{P} \cup \{\top\}$. Do we need $\langle\psi\rangle\phi \to [\psi]\phi$?

The crucial axioms

PR is in the shape of $\langle a\rangle\Diamond\phi \to \Diamond\langle a\rangle\phi$ (or $\Box[a]\phi \to [a]\Box\phi$).

NM is in the shape of $\Diamond\langle a\rangle\phi \to [a]\Diamond\phi$ (or $\langle a\rangle\Box\phi \to \Box[a]\phi$).

No Learning (NL) in ETL: $\Diamond\langle a\rangle\phi \to \langle a\rangle\Diamond\phi$ (or $[a]\Box\phi \to \Box[a]\phi$).
Note the **difference** between NM and NL:

$$\Diamond\langle a\rangle\phi \to [a]\Diamond\phi \text{ (NM) vs. (NL) } \Diamond\langle a\rangle\phi \to \langle a\rangle\Diamond\phi$$

NL is too strong: if you consider possible that an event is executable then it must be executable (take ϕ to be $\top$).

One secret of PAL (and DEL in general) is the *no miracles*-like axiom/property: You can only learn by observation (based on the executability of the actions). It also causes technical difficulties.

Lemma

For all $\phi \in \mathsf{PAL}$*:* $\mathbb{M} \vDash \phi \implies \mathbb{C} \Vdash \phi$.

Lemma

For all $\phi \in \mathsf{PAL}$*:* $\mathbb{C} \Vdash \phi \iff \vdash_{PAN} \phi$.

Theorem

PAN is sound and strongly complete w.r.t. the standard semantics of PAL *on the class of all Kripke frames.*

The proof strategy consists of:

1. Establish the equivalence between the standard semantics and the two-dimensional semantics on ETL-like models within a special class.
2. Axiomatize the ETL-like logic.

Flatten the dynamics!

This can be viewed as another kind of reduction in general which does not eliminate the dynamic modality.

Now we can understand the reduction axioms better!

Axiom Schemas	
TAUT	all the instances of tautologies
DISTK	$K_i(\phi \to \psi) \to (K_i\phi \to K_i\psi)$
!ATOM	$[\psi]p \leftrightarrow (\psi \to p)$
!NEG	$[\psi]\neg\phi \leftrightarrow (\psi \to \neg[\psi]\phi)$
!CON	$[\psi](\phi \wedge \chi) \leftrightarrow ([\psi]\phi \wedge [\psi]\chi)$
!K	$[\psi]\Box\phi \leftrightarrow (\psi \to \Box(\psi \to [\psi]\phi))$

Note: each instance of $\langle\psi\rangle\phi \to [\psi]\phi$ is provable in **PAN**.

The reduction is fragile: what if the updates change the valuation and do not have functionality? It does not matter at all! See e.g., [Wang and Li, 2012].

In terms of logic (valid formulas), PAL (and DEL) are just special ETL-like logics.

Our axiomatization can help to explain many recent results about PAL or other dynamic epistemic logics:

- An explanation to the “reduction phenomena”.
- The axiomatization of the “substitution core” of PAL as in [Holliday et al., 2012].
- The representation results between action model DEL and ETL as in [van Benthem et al., 2009] and [Dégremont et al., 2011].
- The characterization result of partial p-morphism as in [van Benthem, 2012].

The distinction between ETL and DEL is more about different perspectives in semantics: local vs. global. What kind of global properties can be constructed by local constructions?

There are also many new questions

- Can you axiomatize the substitution core of PAL(the collection of valid formulas which are closed under uniform substitution)? [Holliday et al., 2012]
- Can you characterize (syntactically) the “successful” fragment of PAL? [Holliday and III, 2010]
- What operations can be defined by reduction axioms? [van Benthem, 2012]
- Three-value semantics of PAL. [Dechesne et al., 2008]

PAL with natural extensions:

- Quantifying over announcements. [Ågotnes et al., 2009]
- PAL with protocols. [**?**]
- PAL with agent types. [Liu and Wang, 2013]
- With common knowledge: more expressive than modal logic [van Benthem et al., 2006]. A expressiveness hierarchy: [Zou, 2012]
- With iterations: undecidable on the class of arbitrary models [Moss and Miller, 2005], but decidable on single-agent S5. [Ding, 2014]

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