Epistemic Logic (IX)

Logics of knowing whether and much more

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Background

Understanding the expressivity

Axiomatizations

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Happiness and knowledge

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Sometimes, it is better not to know...

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Given a single fact p in the 2-agent setting, how many different epistemic states (maximal consistent sets in $\mathbb{S}5$-**EL**) are there?

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- In particular, $K_2 \neg K_1 \neg K_2 K_1 p$ and $\neg K_2 K_1 p$ are inconsistent.

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(2)	$K_1\neg K_1p \to K_1\neg K_2K_1p$	$(\mathtt{NEC}, \mathtt{DIST})$
(3)	$\neg K_1p \to K_1\neg K_1p$	$(\mathtt{5})$
(4)	$\neg K_1p \to K_1\neg K_2K_1p$	$(\mathtt{MP}(2)(3))$
(5)	$\neg K_1\neg K_2K_1p \to K_1p$	
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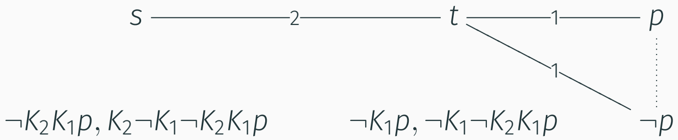

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- $\phi_0 = \neg p$; $\phi_1 = p$
- if $|s| > 0$ is odd: $\phi_{sx} = \begin{cases} \neg Kw_1\phi_s & x = 0 \\ Kw_1\phi_s & x = 1 \end{cases}$
- if $|s| > 0$ is even: $\phi_{sx} = \begin{cases} \neg Kw_2\phi_s & x = 0 \\ Kw_2\phi_s & x = 1 \end{cases}$

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We can show
$\Phi_w = \{\phi_s \mid s \text{ is an non-empty initial segment of } w\}$ are all consistent for each $w \in \{0,1\}^\omega$.

Let w_k be the kth position of w, e.g., $(01\ldots)_1 = 0$, $(0010\ldots)_3 = 1$.

Build a "canonical" model: $\mathcal{M} = \langle \Phi_w \mid w \in \{0,1\}^\omega, \sim_i, V \rangle$ where

- $\Phi_w \sim_1 \Phi_v$ iff $w_k = v_k$ for all the even k and if $w_k = v_k = 1$ for some even k then $w_{k-1} = v_{k-1}$.
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Key observation: $\vDash Kw_i\phi \leftrightarrow Kw_i\neg\phi$ and the rule of replacement for equals for Kw_i: if $w = 0001...$ you want to show $\neg Kw_2\neg Kw_1\neg p$ holds on all the worlds $\Phi_v \sim_1 \Phi_w$ (equiv. $\neg Kw_2 Kw_1 p$, $\neg Kw_2 Kw_1 \neg p$).

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Fragments of **EL** [Parikh and Krasucki 92]; more generally see Klein and Pacuit (manuscript, presented at LOFT).

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Using *Kw* can give us (exponential) succinctness (try to unravel ϕ_S we had before).

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- deontic ∇: moral indifference [von Wright 51]
- proof theoretical ∇: undecided [Zolin 2001]

Understanding the expressivity

Non-contingency (knowing whether) operator

NCL is defined as follows:

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$$\mathcal{M}, s \vDash \Delta_i \phi \quad \Leftrightarrow \quad \text{for all } t_1, t_2 \text{ such that } s \to_i t_1, s \to_i t_2 : (\mathcal{M}, t_1 \vDash \phi \Leftrightarrow \mathcal{M}, t_2 \vDash \phi)$$

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	$\Leftrightarrow$	either for all t such that $s \rightarrow_i t : \mathcal{M}, t \vDash \phi$ or for all t such that $s \rightarrow_i t : \mathcal{M}, t \nvDash \phi$

NCL is clearly no more expressive than ML since we can define a translation $t : \mathsf{NCL} \to \mathsf{ML}$ such that:

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Are they equally expressive over arbitrary models? If not, how to characterize the expressive power of NCL within ML?

Standard bisimilarity is too strong for NCL

Definition (Standard Bisimulation)

Let $\mathcal{M} = \langle S, \{\rightarrow_i \mid i \in \mathrm{I}\}, V \rangle$, $\mathcal{N} = \langle S', \{\rightarrow'_i \mid i \in \mathrm{I}\}, V' \rangle$ be two models. A binary relation Z over $S \times S'$ is a *bisimulation* between $\mathcal{M}$ and $\mathcal{N}$, if Z is non-empty and whenever sZs':

- (Invariance) s and s' satisfy the same propositional variables;
- (Zig) if $s \rightarrow_i t$, then there is a t' such that $s' \rightarrow_i t'$ and tZt';
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$\mathcal{M}, s$ is *bisimilar* to $\mathcal{N}, t$ ($\mathcal{M}, s \leftrightarrows \mathcal{N}, t$) if there is a bisimulation between $\mathcal{M}$ and $\mathcal{N}$ linking s with t.

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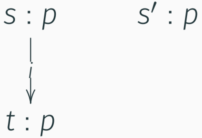
Standard bisimilarity is clearly an invariance relation for `NCL` but it is too strong:



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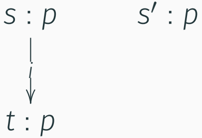

These two finite models satisfy the same NCL formulas but they are clearly not bisimilar.

But in most of the cases when there are two and more successors the standard bisimulation seems fine.

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This inspires us to:

- come up with a structural equivalence notion of Δ_i-*bisimulation* and characterize the expressive power;
- come up with the right definition of canonical relations in the latter completeness proofs;
- find the right axioms for special frame properties.

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Definition (Δ-Bisimulation)

Let $\mathcal{M} = \langle S, R, V \rangle$ be a model. A binary relation Z over S is a Δ-*bisimulation* on $\mathcal{M}$, if Z is non-empty and whenever sZs':

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- (Zag) if there are two different successors t'_1, t'_2 of s' such that $(t'_1, t'_2) \notin Z$ and $s' \to_i t'$, then $\exists\, t$ s.t. $s \to_i t$ and tZt'.

Δ-Bisimilarity

$\mathcal{M}, s$ and $\mathcal{N}, t$ are Δ-bisimilar ($\mathcal{M}, s \leftrightarroweq_{\Delta} \mathcal{N}, t$) if there is a Δ-bisimulation on the *disjoint union* of $\mathcal{M}$ and $\mathcal{N}$ linking s and t.

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*For image-finite (or **NCL** saturated models) $\mathcal{M}, s$ and $\mathcal{N}, t$:*
$\mathcal{M}, s \leftrightarroweq_\Delta \mathcal{N}, t \iff \mathcal{M}, s \equiv_{\mathbf{NCL}} \mathcal{N}, t$ *(satisfying the same **NCL** formulas).*

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The use of Δ-bisimilarity: model expressivity

Theorem (Fan, Wang, van Ditmarsch AiML14)

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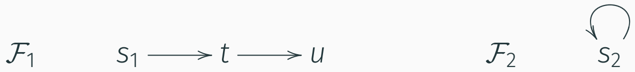

We can show $\mathcal{F}_1 \vDash \phi \iff \mathcal{F}_2 \vDash \phi$ by its contrapositive and $\leftrightarroweq_{\Delta}$.
Then towards contradiction…

Axiomatizations

Axiomatizations of NCL over various frame classes

Apparent difficulties:

- NCL formulas cannot capture the frame properties.
- NCL is not normal:
  - $\Delta_i(\phi \rightarrow \psi) \wedge \Delta_i\phi \rightarrow \Delta_i\psi$ is not valid.
  - $\Delta(\phi \wedge \psi) \rightarrow (\Delta\phi \wedge \Delta\psi)$ is not valid.
  - Monotonicity rule is not valid.
- With extra axioms like $\Delta_i\phi \leftrightarrow \Delta\neg\phi$.

Consider the following axiom schemas and rules as system $\mathbb{SNCL}$:

`TAUT`	all instances of tautologies
`KwCon`	$\Delta_i\phi \wedge \Delta_i\psi \to \Delta_i(\phi \wedge \psi)$
`KwDis`	$\Delta_i\phi \to \Delta_i(\phi \to \psi) \vee \Delta_i(\neg\phi \to \chi)$
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`KwTop`	$\Delta_i\top$
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$\mathbb{SNCL}$ *is sound and strongly complete w.r.t.* ***NCL*** *over the class of arbitrary frames (and serial frames).*

The proof is based on the following canonical model construction, inspired by the "almost definability" schema AD:

$$\neg\Delta_i\psi \rightarrow (\Box_i\phi \leftrightarrow (\Delta_i\phi \wedge \Delta_i(\psi \rightarrow \phi)))$$

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Define $\mathcal{M}^c = \langle S^c, R^c, V^c \rangle$ as follows:

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See the similarity with the standard canonical definition:

For all $s, t \in S^c$, $sR^c_i t$ iff for all ϕ: $\Box\phi \in s$ implies $\phi \in t$.

Truth lemma

Lemma

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1. $\{\phi \mid \Delta_i\phi \wedge \Delta_i(\psi \to \phi) \in s\} \cup \{\psi\}$ is consistent.
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1 relies on the validity of the following long formula (proved by using `KwCon` and `KwDis`.) and 2 needs `KwNeg` in addition.

$$\Delta_i(\bigwedge_{j=1}^{k} \phi_j \to \neg\psi) \land \bigwedge_{j=1}^{k} \Delta_i\phi_j \land \bigwedge_{j=1}^{k} \Delta_i(\psi \to \phi_j) \to \Delta_i\psi$$

NCL over other frame classes

Notation	Axiom Schemas	Systems	Frames
KwT	$\Delta_i\phi \wedge \Delta_i(\phi \to \psi) \wedge \phi \to \Delta_i\psi$	$\mathbb{SNCLT} = \mathbb{SNCL} + \texttt{KwT}$	*reflexive*
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- We find axioms inspired by AD.
- We manipulate the canonical model.

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Simplify the semantics while keeping the logic

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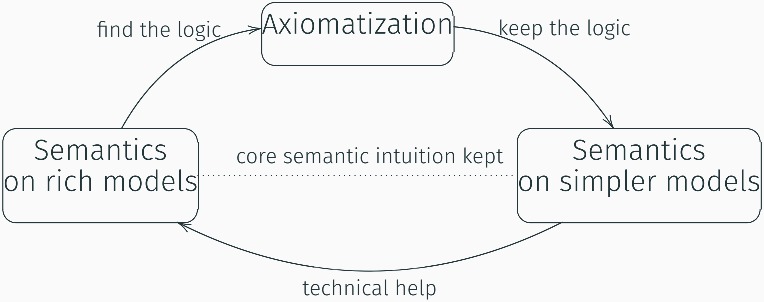
To restore the balance between the language and model:



Recall the axiomatization

`TAUT`	all instances of tautologies
`KwCon`	$\Delta_i(\phi) \wedge \Delta_i(\psi) \to \Delta_i(\phi \wedge \psi)$
`KwDis`	$\Delta_i\phi \to \Delta_i(\phi \to \psi) \vee \Delta_i(\neg\phi \to \chi)$
`KwNeg`	$\Delta_i\phi \leftrightarrow \Delta_i\neg\phi$
`KwTop`	$\Delta_i\top$
`MP`	From ϕ and $\phi \to \psi$ infer ψ
`REKw`	From $\phi \leftrightarrow \psi$ infer $\Delta_i\phi \leftrightarrow \Delta_i\psi$

Alternative neighbourhood semantics [Fan SL 17]

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The semantics is given by:

$$\mathcal{M}, w \Vdash \Box\phi \iff [\![\phi]\!]^{\mathcal{M}} \in N(w)$$

We say the neighbourhood model is a **NCL**-model if for all $w \in W$

- $W \in N(w)$
- $N(w)$ is closed under complementation
- $N(w)$ is closed under intersection
- $N(w)$ is closed under supersets or co-supersets: $X, Y, Z \subseteq W, X \in N(w)$ implies $X \cup Y \in N(w)$ or $(W \setminus X) \cup Z \in N(w)$.

We have exactly the same valid formulas over **NCL**-models.

Dynamics

We can add public announcements and event updates into the language PALNC:

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With the usual reduction axiom and the following one we can easily axiomatize **PALNC** over various classes of frames:

$$[\phi]\Delta_i\psi \leftrightarrow (\phi \rightarrow (\Delta_i[\phi]\psi \vee \Delta_i[\phi]\neg\psi))$$

Similar story holds if we introduce the event modality.

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What is the general theory of such propositional bundles?

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A related question: ultimate ignorance $\bigwedge_{s\in G^*} \nabla_s\phi$. It is also the ultimate independence in the setting of probability logic.

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Surprisingly, the two communities were ignorant about each other's work on such logics! See our RSL article.