Epistemic Logic

III Some (philosophical) problems of the basic framework

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Recap: basic systems

Epistemic Language (EL) and semantics

The Epistemic Language:

$$\phi \quad ::= \quad \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid K_i\phi$$

It is interpreted on Kripke models $\mathcal{M} = (S, \{\to_i\}_{i\in I}, V)$ where $\to_i$ has certain properties (sometimes taken as an equivalence relation).

$$\boxed{\mathcal{M}, s \models K_i\phi \quad \Leftrightarrow \quad \forall t : s \to_i t \implies \mathcal{M}, t \models \phi}$$

$S5$ system (strongest epistemic logic)

System $S5$

Axioms		Rules	
TAUT	all the instances of tautologies	MP	$\frac{\phi, \phi \to \psi}{\psi}$
DISTK	$K_i(p \to q) \to (K_i p \to K_i q)$	NECK	$\frac{\phi}{K_i \phi}$
T	$K_i p \to p$	SUB	$\frac{\phi}{\phi[p/\psi]}$
4	$K_i p \to K_i K_i p$		
5	$\neg K_i p \to K_i \neg K_i p$		

Note that S5=KT5=KTB4 modulo theorems.

$S5$ is sound and strongly complete for modal logic over frames with equivalence relations.

*KD*45 system (strongest doxastic logic)

System *KD*45

Axioms		Rules	
TAUT	all the instances of tautologies	MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
DISTK	$B_i(p \to q) \to (B_i p \to B_i q)$	NECK	$\dfrac{\phi}{B_i \phi}$
D	$\neg B_i \bot$ (or $B_i p \to \neg B_i \neg p$)	SUB	$\dfrac{\phi}{\phi[p/\psi]}$
4	$B_i p \to B_i B_i p$		
5	$\neg B_i p \to B_i \neg B_i p$		

*KD*45 is sound and strongly complete for modal logic over frames that are transitive, euclidean and serial.

Moore sentence and Church-Fitch (knowability) paradox

$$\vdash_{KT} K_i(p \land \neg K_i p) \rightarrow \bot$$

$$\vdash_{KD4} B_i(p \land \neg B_i p) \rightarrow \bot$$

Verificationist's theory of truth requires $\phi \rightarrow \Diamond K\phi$

$(p \land \neg Kp) \land \neg\Diamond K(p \land \neg Kp)$ under the condition that $\neg\Diamond\bot$.

Requiring $\phi \rightarrow \Diamond K\phi$ for all ϕ will result in $\phi \rightarrow K\phi$.

(S5) knowledge = true (KD45) belief?

KD45 agents thought they are S5 agents!

KD45 belief plus the following definition:

$$K_i\phi \equiv_{df} \phi \wedge B_i\phi$$

Does it give us the S5 axioms of K_i? Negative introspection does not work!

A basic system combing knowledge and belief (Stalnaker 06)

Consider the system of KD belief and S4 knowledge plus some interaction axioms:

- $B(p \to q) \to (Bp \to Bq)$
- $\neg B\bot$
- $K(p \to q) \to (Kp \to Kq)$
- $Kp \to KKp$
- $Kp \to p$
- $Kp \to Bp$
- $Bp \to KBp$
- $\neg Bp \to K\neg Bp$
- $Bp \to BKp$ (strong belief)

B is actually $KD45$ and we can show that $Bp \leftrightarrow \neg K\neg Kp$. Then we have Geach theorem (.2) $\neg K\neg Kp \to K\neg K\neg p$ from D for K.

Introspections

Hintikka's arguments

On positive introspection:

- If $\{K\phi, \neg K\neg\psi\}$ is consistent
- Then $\{K\phi, \psi\}$ is consistent
- Substitute ψ above with $\neg K\phi$ we will have the KK principle (Axiom 4).

On negative introspection:

- Assuming 5, we have axiom $B : p \to K\neg K\neg p$.
- This seems to rule out the possibility of false beliefs.

Negative introspection

$$\neg K_i p \rightarrow K_i \neg K_i p$$

Lenzen (1978)'s example:

- Suppose $B_i K_i p$ but $\neg p$: *i* falsely believes that he knows *p*.
- then $\neg K_i p$ (**T** axiom)
- therefore $K_i \neg K_i p$ (**5** axiom)
- thus $B_i \neg K_i p$ (knowledge implies belief)
- conclude $B_i \bot$ (under normal axioms for B_i)

We cannot express false belief of knowledge!

The equivalent form: $\neg K_i \neg K_i p \rightarrow K_i p$ sounds more problematic: if you "think" it is possible to know then you know it.

Lenzen prefers S4.2: $\neg K_i \neg K_i p \rightarrow K_i \neg K_i \neg p$

Positive introspection

Williamson (1992, 2000) on inexact knowledge (assuming axiom 4 and T) where p_k means a tree is k cm tall (simplified version):

1. assuming (for all reasonable k): $K(p_{k+1} \rightarrow \neg K \neg p_k)$
2. assuming $K \neg p_0$
3. by (2) and axiom 4: $KK \neg p_0$
4. by (1) and `NEC` and `DIST` we have $KK \neg p_0 \rightarrow K \neg p_1$
5. by (3) and (4): $K \neg p_1$
6. repeat to derive $K \neg p_k$
7. Suppose p_{666} is true then it contradicts to $K \neg p_{666}$ (under `T` axiom)

(1) is fishy: From (1) we can derive $p_{k+1} \rightarrow \neg K \neg p_k$. Moreover, by T axiom $Kp_{k+1} \rightarrow p_{k+1}$. Hence, we can infer $Kp_{k+1} \rightarrow \neg K \neg p_k$.

Transitivity

Transitivity is not that reasonable especially in the setting of inexact knowledge, e.g., a spectrum of redness.

Can we have positive introspection without transitivity? Yes…

Logical omniscience

Logical omniscience

Problematic closure rules:

- From $\vdash \phi$ infer $K\phi$
- From $\vdash \phi \rightarrow \psi$ infer $K\phi \rightarrow K\psi$
- From $\vdash \phi \leftrightarrow \psi$ infer $K\phi \leftrightarrow K\psi$

'Deductive closure principle': $K(\phi \rightarrow \psi) \wedge K\phi \rightarrow K\psi$ is a different thing (which can also be challenged though, e.g., examples by Dretske (1970)).

The problem is more about the complexity of the logical reasoning!

Logical omniscience

Ideas of the 'solutions':

- Reinterpretation: implicit knowledge; of an ideal agent
- Syntactic approach: a set of formulas
- Impossible worlds: inconsistent alternatives
- Awareness: K=awareness+implicit knowledge
- Algorithmic knowledge: K=answer by algorithm
- Neighbourhood semantics: still problematic (From $\vdash \phi \leftrightarrow \psi$ infer $K\phi \leftrightarrow K\psi$)
- Timed knowledge: reasoning takes time
- but, can we do better? Yes, we can!

External vs. Internal

Modeller or modelled

- Help to model other's reasoning
- Help us to reason by ourselves

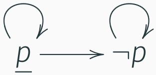
However, can the agents 'see' the model (without the real world)?

It seems OK for S5 models, what about KD45?

For the KD45 agent the model is actually:

Is it good enough to have generated submodel based on the states which are reachable from the real world?

Now consider the following S4 model (reflexive and transitive):



If the agent can see the model (without the real world) himself, then he can reason as follows (suggested by Yang Liu):

I am not considering $\neg p$ as the only possibility, thus I must be on a p world, then I should know p! What is wrong?

At the p world: $K(\neg p \to K\neg p)$ is true thus $K\neg K\neg p \to Kp$. Note that $\neg K\neg p$ is also true at p but we don't have $K\neg K\neg p$ since it is S4 and you are not allowed to use negative introspection!

By having the model, you are implicitly using negative introspection: if I don't know $\neg p$ then I know I don't know it. The meta reasoning violates the assumption (the model is your brain).

Group notions

Common knowledge

David Lewis' *Convention* (1969): $p :=$ every driver must drive on the right.
What kind of knowledge is enough to let people feel safe in driving on the right? Is 'everybody knows that p' enough (Ep)?
What about everybody knows that everybody knows that p (EEp)?

No, since agent i considers possible that agent j considers possible that agent i does not know ($\neg K_i K_j K_i p$) and thus agent j may drive on the left. You may argue by induction that $E^k p$ is not enough.

$$C\phi := E\phi \wedge EE\phi \wedge EEE\phi....$$

$$\boxed{\begin{array}{lcl} \mathcal{M}, s \models E_G\phi & \Leftrightarrow & \text{for all } t \text{ such that } s \xrightarrow{EG} t, \mathcal{M}, t \models \phi \\ \mathcal{M}, s \models C_G\phi & \Leftrightarrow & \text{for all } t \text{ such that } s \xrightarrow{CG} t, \mathcal{M}, t \models \phi \end{array}}$$

where $\xrightarrow{EG} = \bigcup_{i \in G} \xrightarrow{i}$ and $\xrightarrow{CG} = (\xrightarrow{EG})^*$ (the reflexive transitive closure of $\xrightarrow{EG}$).

Axiomatization:

- S5 system for K plus `NEC` and `DIST` for C_G and
- Fixed-Point Axiom: $C_G\phi \leftrightarrow (\phi \wedge E_G C_G \phi)$
- Induction Axiom: $(\phi \wedge C_G(\phi \rightarrow E_G\phi)) \rightarrow C_G\phi$ or
- Induction *Rule*: $\vdash \psi \rightarrow E_G(\psi \wedge \phi)$ infer $\vdash \psi \rightarrow C_G\phi$

Common knowledge

Questions:

- Can we easily have full common knowledge? e.g., consecutive numbers.
- Do we really need the full power of common knowledge in many cases? tricky case: muddy children.
- What if I don't know who are there in the group? Common knowledge w.r.t. "group agent" without the explicit set of agents.

Distributed knowledge

Intuition: what we know if we put all of our knowledge together.

$$\boxed{\mathcal{M}, s \models D_G\phi \quad \Leftrightarrow \quad \text{for all } t \text{ such that } s \stackrel{DG}{\to} t, \mathcal{M}, t \models \phi}$$

where $\stackrel{DG}{\to} = \bigcap_{i \in G} \stackrel{i}{\to}$.

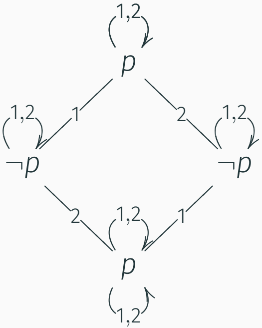
Finite axiomatization (a wise man's knowledge):

- S5 axioms for D_G plus
- $K_i\phi \to D_G\phi$ (when $i \in G$)

Distributed knowledge

Questions:

- It is not invariant under bisimulation;
- It is not the case: $\{\psi \mid \mathcal{M}, s \vDash K_i\psi \text{ for some } i \in G\} \vDash \phi$ iff $\mathcal{M}, s \vDash D_i\phi$ for all ϕ.



is bisimilar to

Other problems

- Modeling vs. model hecking
- Justification
- Agency
- Understanding
- Other notions of knowing: correlation, procedure