Neighborhood Semantics and Topological Semantics for Modal Logic

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2020.5.19

Augmentation

Definition (Augmented I)

A minimal model $\mathcal{U} = \langle W, N, P \rangle$ is augmented if and only if it is supplemented and, for every world α in it,

$$\bigcap N_\alpha \in N_\alpha$$

Supplemented: (m) if $X \cap Y \in N_\alpha$, then $X \in N_\alpha$ and $Y \in N_\alpha$.
Supplemented: (m') if $X \subseteq Y$ and $X \in N_\alpha$, then $Y \in N_\alpha$.

Definition (Augmented II)

A minimal model $\mathcal{U} = \langle W, N, P \rangle$ is augmented if and only if for every α and X in $\mathcal{U}$,

$$X \in N_\alpha \leftrightarrow \bigcap N_\alpha \subseteq X$$

Augmentation

Proof.

$(I \Rightarrow II)$ Assume $\mathcal{U} = \langle W, N, P \rangle$ is augmented, we prove that $X \in N_\alpha \leftrightarrow \bigcap N_\alpha \subseteq X$. If $X \in N_\alpha$, then obviously we have $\bigcap N_\alpha \subseteq X$. If $\bigcap N_\alpha \subseteq X$, since $\mathcal{U}$ is augmented so $\bigcap N_\alpha \in N_\alpha$. Then by supplementation we have $X \in N_\alpha$.

$(I \Leftarrow II)$ Assume $\mathcal{U}$ satisfies $X \in N_\alpha \leftrightarrow \bigcap N_\alpha \subseteq X$. We prove that $\mathcal{U}$ is augmented. If $X \subseteq Y$ and $X \in N_\alpha$, then we have $\bigcap N_\alpha \subseteq X$. So $\bigcap N_\alpha \subseteq Y$ and then $Y \in N_\alpha$. So $\mathcal{U}$ is supplemented. And since $\bigcap N_\alpha \subseteq \bigcap N_\alpha$ so $\bigcap N_\alpha \in N_\alpha$. Hence $\mathcal{U}$ is augmented. □

Augmentation

Remark

An augmented model always contains the unit, i.e. N_α *always contains* W *(since we always have* $\bigcap N_\alpha \subseteq W$*).*

Remark

The condition that $\bigcap N_\alpha \in N_\alpha$ *may be described as closure under arbitrary intersections.*

Remark

Every augmented model is a filter: supplemented, closed under intersections, and possessed of the unit.

Remark

Every finite filter is augmented.

Augmentation

Not every filter is augmented!!!

Example

Consider a minimal model $\mathcal{U} = \langle W, N, P \rangle$ in which W is the set of real numbers and, for each real number α,

$$N_\alpha = \{X \subseteq W : (\alpha, \beta) \subseteq X \text{ for some } \beta \in W \text{ such that } \alpha \leq \beta\}$$

We can see that $\mathcal{U}$ is supplemented and contains the unit. We can also show that $\mathcal{U}$ is closed under finite intersections but it is not closed under infinite intersections. Since $\bigcap N_\alpha = \emptyset \notin N_\alpha$ for any $\alpha \in \mathcal{U}$. Hence $\mathcal{U}$ is a filter but not augmented.

Augmentation

Theorem

Every standard model $\mathcal{U}^s = \langle W, R, P \rangle$ has a pointwise equivalent augmented minimal model $\mathcal{U}^m = \langle W, N, P \rangle$, and vice versa.

Proof.

($\Rightarrow$) Let $\mathcal{U}^s$ be a standard model and define the minimal model $\mathcal{U}^m$ by stipulating that

$$X \in N_\alpha \text{ iff } \{\beta \in W : \alpha R \beta\} \subseteq X$$

for every $\alpha \in W$ and every $X \subseteq W$.Then $\bigcap N_\alpha = \{\beta \in W : \alpha R \beta\}$ for each $\alpha \in W$. So $\mathcal{U}^m$ satisfies $X \in N_\alpha$ iff $\bigcap N_\alpha \subseteq X$, which means that it is augmented. The proof that $\mathcal{U}^s$ and $\mathcal{U}^m$ are pointwise equivalent, i.e. that a world verifies the same sentences in each model, is by induction on the complexity of a sentence A.

□

Augmentation

Proof.

The only case of interest is that in which A is a necessitation, $\Box B$, where the argument goes like this:

$$\begin{aligned}
\models_{\alpha}^{\mathcal{U}^s} \Box B & \text{ iff for every } \beta \in W \text{ such that } \alpha R \beta, \models_{\beta}^{\mathcal{U}^s} B \\
& \text{ iff } \{\beta \in W : \alpha R \beta\} \subseteq \|B\|^{\mathcal{U}^s} \\
& \text{ iff } \|B\|^{\mathcal{U}^s} \in N_\alpha \text{ (by definition of } N_\alpha) \\
& \text{ iff } \|B\|^{\mathcal{U}^m} \in N_\alpha \text{ (by IH)} \\
& \text{ iff } \models_{\alpha}^{\mathcal{U}^m} \Box B
\end{aligned}$$

□

Augmentation

Proof.

($\Leftarrow$) let $\mathcal{U}^m$ be an augmented minimal model and define the standard model $\mathcal{U}^s$ by: $\alpha R \beta$ iff $\beta \in \bigcap N_\alpha$ for every α and β in W. As before, necessitation is the only case of interest in the inductive proof that the models are pointwise equivalent. The argument proceeds as follows.

$$
\begin{aligned}
\models^{\mathcal{U}^m}_\alpha \Box B \text{ iff } & \|B\|^{\mathcal{U}^m} \in N_\alpha \\
\text{iff } & \bigcap N_\alpha \subseteq \|B\|^{\mathcal{U}^m} \text{ (because } \mathcal{U}^m \text{ is augmented)} \\
\text{iff } & \bigcap N_\alpha \subseteq \|B\|^{\mathcal{U}^s} \text{ (IH)} \\
\text{iff } & \text{for every } \beta \in W \text{ such that } \alpha R \beta, \models^{\mathcal{U}^s}_\beta B \text{ (by definition of } R) \\
\text{iff } & \models^{\mathcal{U}^s}_\alpha \Box B
\end{aligned}
$$

Augmentation

Now we can define the operation of augmentation, which turns a minimal model into an augmented model.

Definition

Let $\mathcal{U} = \langle W, N, P\rangle$ be a minimal model. The augmentation of $\mathcal{U}$ is the minimal model $\mathcal{U}^! = \langle W, N^!, P\rangle$ in which, for each $\alpha \in W$,

$$N^!_\alpha = \{X \subseteq W : \bigcap N_\alpha \subseteq X\}$$

Remark

$\mathcal{U}^!$ is the supplementation of $\mathcal{U}$ closed under intersection. An augmented model is thus a minimal model identical with its own augmentation.

The schemas $D, T, B, 4, 5$

We consider the following schemas.

$$
\begin{aligned}
D.&\quad \Box A \rightarrow \Diamond A \\
T.&\quad \Box A \rightarrow A \\
B.&\quad A \rightarrow \Box\Diamond A \\
4.&\quad \Box A \rightarrow \Box\Box A \\
5.&\quad \Diamond A \rightarrow \Box\Diamond A
\end{aligned}
$$

None of these is valid in the class of all minimal models. But for each of these schemas we can identify a class of minimal models that validates it. In standard model K, we add some constraints on R to validate $D, T, B, 4, 5$. Similarly, in minimal models, we can also add some constraints on N to validate $D, T, B, 4, 5$.

The schemas $D, T, B, 4, 5$

We wish to consider the following conditions on a minimal model $\mathcal{U} = \langle W, N, P \rangle$, for every world α and proposition X in $\mathcal{U}$:

$$
\begin{aligned}
(d) &\quad \text{if } X \in N_\alpha, \text{then } W \backslash X \notin N_\alpha \\
(t) &\quad \text{if } X \in N_\alpha, \text{then } \alpha \in X \\
(b) &\quad \text{if } \alpha \in X, \text{then } \{\beta \text{ in } \mathcal{U} : W \backslash X \notin N_\beta\} \in N_\alpha \\
(iv) &\quad \text{if } X \in N_\alpha, \text{then } \{\beta \text{ in } \mathcal{U} : X \in N_\beta\} \in N_\alpha \\
(v) &\quad \text{if } X \notin N_\alpha, \text{then } \{\beta \text{ in } \mathcal{U} : X \notin N_\beta\} \in N_\alpha
\end{aligned}
$$

The schemas $D, T, B, 4, 5$

Theorem

The following schemas are valid respectively in the indicated classes of minimal models.

$$\begin{aligned} &(1)\quad D : \textit{condition } (d) \\ &(2)\quad T : \textit{condition } (t) \\ &(3)\quad B : \textit{condition } (b) \\ &(4)\quad 4 : \textit{condition } (iv) \\ &(5)\quad 5 : \textit{condition } (v) \end{aligned}$$

The schemas $D, T, B, 4, 5$

Proof.

Let α be a world in a minimal model $\mathcal{U} = \langle W, N, P \rangle$.
For (1). Suppose $\mathcal{U}$ satisfies (d), and that $\Box A$ is true at α.Then $\|A\|^{\mathcal{U}} \in N_\alpha$,and so by (d), $W \backslash \|A\|^{\mathcal{U}} \notin N_\alpha$,which means that $\Diamond A$ is true at α.It follows that D is valid in the class of minimal models that satisfy condition (d).
For (3). Here we suppose that $\mathcal{U}$ satisfies condition (b), and that A is true at α.In other words, $\alpha \in \|A\|^{\mathcal{U}}$, from which it follows by (b) that $\{\beta \in \mathcal{U} : W \backslash \|A\|^{\mathcal{U}} \notin N_\beta\} \in N_\alpha$.This means that $\{\beta \in \mathcal{U} : \vDash^{\mathcal{U}}_{\beta} \Diamond A\} \in N_\alpha$,i.e. that $\|\Diamond A\|^{\mathcal{U}} \in N_\alpha$.But this last just means that $\Box\Diamond A$ is true at α, which is what we needed to show.Thus the schema B is valid in the class of minimal models satisfying condition (b). □

Topological Models

Much of the original motivation for neighborhood structures as a semantics for modal logic comes from elementary point-set topology. In this section, We discuss topological semantics for modal logic. The idea to interpret the basic modal language on topological models is usually attributed to McKinsey and Tarksi.

Definition (Topological Space)

A topological space is a subset space $\langle W, \mathcal{T} \rangle$, where W is a nonempty set and $\mathcal{T} \subseteq \wp(W)$ satisfying

- 1. $W \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$;
- 2. $\mathcal{T}$ is closed under finite intersections; and
- 3. $\mathcal{T}$ is closed under arbitrary unions.

Elements $O \in \mathcal{T}$ are called opens. A set C such that $W \backslash C \in \mathcal{T}$ is said to be closed.

Topological Models

Suppose that $\langle W, \mathcal{T} \rangle$ is a topological space and $X \subseteq W$ is any set. The largest open subset of X is called the interior of X, denoted $Int(X)$. Formally,

$$Int(X) = \bigcup\{O | O \in \mathcal{T} \text{ and } O \subseteq X\}$$

The smallest closed set containing X is called the closure of X, denoted $Cl(X)$. Formally,

$$Cl(X) = \bigcap\{C | W \backslash C \in \mathcal{T} \text{ and } X \subseteq C\}.$$

It is easy to see that a set X is open if $Int(X) = X$ and closed if $Cl(X) = X$.

Topological Models

Lemma

Let $\langle W, \mathcal{T} \rangle$ be a topological space and $X, Y \subseteq W$. Then,

- *1. $Int(X \cap Y) = Int(X) \cap Int(Y)$.*
- *2. $Int(\emptyset) = \emptyset$, $Int(W) = W$.*
- *3. $Int(X) \subseteq X$.*
- *4. $Int(Int(X)) = Int(X)$.*

Lemma

Let $\langle W, \mathcal{T} \rangle$ be a topological space and $X, Y \subseteq W$. Then,

- *1. $Cl(X \cup Y) = Cl(X) \cup Cl(Y)$.*
- *2. $Cl(\emptyset) = \emptyset$, $Cl(W) = W$.*
- *3. $X \subseteq Cl(X)$.*
- *4. $Cl(Cl(X)) = Cl(X)$.*

Topological Models

Every topological space $\langle W, \mathcal{T} \rangle$ defines an interior operator $Int : \wp(W) \to \wp(W)$ and a closure operator $Cl : \wp(W) \to \wp(W)$ satisfying the properties from previous Lemma.

Topological spaces can be used as a semantics for a propositional modal language by interpreting the Boolean connectives in the usual way and interpreting the modalities as operators associated with the topology.For instance, McKinsey and Tarski interpret the box-modality as the interior operator for a topological space.

Topological Models

Definition (Topological Model)

A topological model is a tuple $\langle W, \mathcal{T}, V \rangle$, where $\langle W, \mathcal{T} \rangle$ is a topological space and $V : At \to \wp(W)$ is a valuation function.

Formulas of $\mathcal{L}(At)$ are interpreted at states $w \in W$. The Boolean connectives and atomic propositions are interpreted as usual. The definition of truth for the modal operator is:

$$\mathcal{M}^{\mathcal{T}}, w \models \Box\varphi \text{ iff there is an } O \in \mathcal{T}, \text{ such that } w \in O$$
$$\text{and for all } v \in O \setminus \{w\}, \mathcal{M}^{\mathcal{T}}, v \models \varphi.$$

Recall the notation for the truth set of a formula $\varphi \in \mathcal{L}(At) : \|\varphi\|_{\mathcal{M}^{\mathcal{T}}} = \{w | \mathcal{M}^{\mathcal{T}}, w \models \varphi\}$.It is an immediate consequence of the definitions that for any formula $\varphi \in \mathcal{L}(At)$ and topological model $\mathcal{M}^{\mathcal{T}}$, $\|\Box\varphi\|_{\mathcal{M}^{\mathcal{T}}} = Int(\|\varphi\|_{\mathcal{M}^{\mathcal{T}}})$.

Topological Models

There are other operators associated with topological spaces that can be used as a semantics for a modal operator. One influential approach is to use the derived set operator.

Definition (Limit point)

Suppose that $\langle W, \mathcal{T} \rangle$ is a topological space. A point $w \in W$ is called a limit point of $X \subseteq W$ provided that for each open set $O \in \mathcal{T}$ such that $w \in O$, $X \cap (O \backslash \{w\}) \neq \emptyset$.

The derived set operator is a function $Der : \wp(W) \rightarrow \wp(W)$, where for all $X \subseteq W$, $Der(X) = \{w | w \text{ is a limit point of } X\}$ ($Der(X)$ is also called the derivative of X). The derived set operator is often used as an alternative characterization of closed sets. We can prove that for any set $X \subseteq W$, $Cl(X) = X \cup Der(X)$.

Topological Models

The key idea is to interpret the diamond modality as the derived set operator. Suppose that $\mathcal{M}^{\mathcal{T}} = \langle W, \mathcal{T}, V \rangle$ is a topological model with $w \in W$. The definition of truth for the two modalities is:

$$\mathcal{M}^{\mathcal{T}}, w \models \Box\varphi \text{ iff there is an } O \in \mathcal{T}, \text{ such that } w \in O \text{ and for all } v \in O \backslash \{w\}, \mathcal{M}^{\mathcal{T}}, v \models \varphi.$$

$$\mathcal{M}^{\mathcal{T}}, w \models \Diamond\varphi \text{ iff for all } O \in \mathcal{T} \text{ with } w \in O, \text{ there is a } v \in O \backslash \{w\} \text{ such that } \mathcal{M}^{\mathcal{T}}, v \models \varphi.$$

Then for any formula $\varphi \in \mathcal{L}(At)$ and topological model $\mathcal{M}^{\mathcal{T}}$,

$$\|\Box\varphi\|_{\mathcal{M}^{\mathcal{T}}} = Int(\|\varphi\|_{\mathcal{M}^{\mathcal{T}}})$$
$$\|\Diamond\varphi\|_{\mathcal{M}^{\mathcal{T}}} = Der(\|\varphi\|_{\mathcal{M}^{\mathcal{T}}})$$

Topological Spaces and Relational Structures

There is a well-known connection between relational frames and certain topological spaces.A topological space $\langle W, \mathcal{T} \rangle$ is called an **Alexandroff** space provided that for any (not just finite) $\mathcal{X} \subseteq \mathcal{T}, \bigcap \mathcal{X} \in \mathcal{T}$.An Alexandroff topology has the additional property that arbitrary intersections of open sets are open.
A set $X \subseteq W$ is called an **R-upset**, denoted $X^{\uparrow R}$, provided that $w \in X$ and wRv implies that $v \in X$.The set of R-upsets for a reflexive and transitive relation R forms an Alexandroff topology:

Proposition

Suppose that $\langle W, R \rangle$ is reflexive and transitive relational frame. Let $\langle W, \mathcal{T}_R \rangle$ be a subset space where $\mathcal{T}_R = \{X | X \text{ is an R-upset}\}$. Then $\langle W, \mathcal{T}_R \rangle$ is an Alexandroff topology.

Topological Spaces and Relational Structures

We can also construct an S4-relational frame from a topology. Suppose that $\langle W, \mathcal{T}\rangle$ is a topological space.The **specialization order**, $R_{\mathcal{T}} \subseteq W \times W$, is defined as follows $wR_{\mathcal{T}}v$ iff $v \in Cl(\{w\})$.Thus, $wR_{\mathcal{T}}v$ provided that v is in every closed set that contains w.We can show that $\langle W, R_{\mathcal{T}}\rangle$ is an S4-relational frame. Thus, every topological space $\langle W, \mathcal{T}\rangle$ is associated with an S4-relational frame $\langle W, R_{\mathcal{T}}\rangle$.And there is a much tighter connection when the topology is Alexandroff.

Proposition

Suppose that $\langle W, \mathcal{T}\rangle$ is a topological space. Then

- $\mathcal{T} \subseteq \mathcal{T}_{R_{\mathcal{T}}}$ *; and*
- $\mathcal{T} = \mathcal{T}_{R_{\mathcal{T}}}$ *iff T is Alexandroff.*

Topological Spaces and Neighborhood Structures

Suppose that $\langle W, \mathcal{T}\rangle$ is a topological space. For each $w \in W$, the set of open sets containing w is $\mathcal{T}_w = \{O | O \in \mathcal{T} \text{ and } w \in O\}$.A **neighborhood** (in the topological sense) of a point $w \in W$ is a set X such that there is some $O \in \mathcal{T}_w$ such that $O \subseteq X$.That is, X is a neighborhood of w if X contains an open set containing w.

Definition (Neighborhood System)

Suppose that $\langle W, \mathcal{T}, V\rangle$ is a topology. A neighborhood system for $\mathcal{T}$ is a function $N_{\mathcal{T}} : W \to \wp(\wp(W))$ such that

$$N_{\mathcal{T}}(w) = \{X | \text{there is an } O \in \mathcal{T}_w \text{ such that } O \subseteq X\}$$

Suppose that $\langle W, \mathcal{T}, V\rangle$ is a topological space. We can show that for all $w \in W$, $N_{\mathcal{T}}(w)$ is a consistent filter($\emptyset \notin N_{\mathcal{T}}(w)$), and $w \in \bigcap N_{\mathcal{T}}(w)$.

Topological Spaces and Neighborhood Structures

Notice that that for all $w \in W$, $\mathcal{T}_w \subseteq N_{\mathcal{T}}(w)$. That is, any open set containing w is a neighborhood of w (such a set is called an **open neighborhood**).If X is an open neighborhood of w (i.e., $X \in \mathcal{T}_w$), then X is a neighborhood of all of its elements. Thus, any neighborhood system $N_{\mathcal{T}}$ satisfies the following property:

$$\text{For all } w \in W, \text{ for all } X \in N_{\mathcal{T}}(w), \text{there is a } Y \subseteq X \\ \text{such that for all } v \in Y, Y \in N_{\mathcal{T}}(v).$$

Using Definition of Neighborhood System, we have that every $\langle W, \mathcal{T} \rangle$ is associated with a neighborhood frame $\langle W, N_{\mathcal{T}} \rangle$.

Topological Spaces and Neighborhood Structures

It turns out that a class of neighborhood frames well-known to modal logicians gives rise to topological spaces:

Definition (S4 Neighborhood Frame)

A neighborhood frame $\langle W, N \rangle$ is an **S4 neighborhood frame** provided that N satisfies the following properties. For any $w \in W$:

- 1. $N(w)$ is a consistent filter;
- 2. $w \in \bigcap N(w)$; and
- 3. for any $X \subseteq W$, if $X \in N(w)$, then $\{v | X \in N(v)\} \in N(w)$.

Proposition

Suppose that $\langle W, N \rangle$ is an S4-neighborhood frame. Then, there is a topology $\langle W, \mathcal{T}_N \rangle$ such that for all $w \in W, N(w) = N_{\mathcal{T}_N}(w)$.

Topological Spaces and Neighborhood Structures

Corollary

For each S4 neighborhood model $\mathcal{M}$, there is a topological model $\mathcal{M}^{\mathcal{T}}$ such that for all $\varphi \in \mathcal{L}$, $\|\varphi\|_{\mathcal{M}} = \|\varphi\|^{\mathcal{T}}_{\mathcal{M}}$.

Since every topological model can be viewed as an S4 neighborhood model (i.e.,a neighborhood model that satisfies the properties from Definition of S4 Neighborhood Frame),we can say that the class of topological models is modally equivalent to the class of S4-neighborhood models.

References I

Brian F. Chellas.
Modal logic: An introduction.
Contemporary American Literature, 1980.

Eric Pacuit.
Neighborhood semantics for modal logic.
2017.