Complexity of Modal Logic (Part I)

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May 21, 2020

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Preliminaries

Computability

In the following investigation, we use the computation model of *Turing machine* and assume a proper *presentation* of model and formulas.

Non-deterministic Turing machine: each action has a (finite) range of options. However, the only know way of implementing non-determinism is to simulate it on a deterministic Turing machine and all known implementation requires exponential time to perform.

Complexity

The complexity of a task is measured in terms of *time* (number of steps taken) and *space* (size of the memory employed).

In complexity theory, problems are classified into various classes.

$$P \subseteq NP \subseteq PSPACE \subseteq EXPTIME \subseteq NEXPTIME$$

It is currently unknown whether the inclusions are strict or not.

Complexity

Let n be the length of the input.

A problem belongs to P if there is a polynomial $p(n)$ and a deterministic Turing machine which solves the problem in at most $p(n)$ steps.

A problem belongs to NP if there is a polynomial $p(n)$ and a non-deterministic Turing machine which solves the problem in at most $p(n)$ steps. Equivalently, if an answer of the problem can be verified in polynomial time.

A problem belongs to PSPACE if there is a polynomial $p(n)$ and a deterministic Turing machine which solves the problem after scanning at most $p(n)$ tape squares.

Complexity

A few theoretical observatsions.

- In P-time steps, you can only visit polynomially many memory locations. So P and NP are contained in $PSPACE$.
- With polynomially many memory locations, you can revisit them; however, there is an exponential upper limit, as repeating the same trajectories makes no sense. So $PSPACE$ is contained in $EXPTIME$.
- According to Savitch's Theorem, $PSPACE = NPSPACE$.

Let C be a class of problems. L_1 and L_2 be (abstract) problems.

Polytime Reduction A polynomial time computable function f from L_1 to L_2 is called a *polytime reduction* if $s \in L_1$ iff $f(s) \in L_2$.

Hardness A problem is *C-hard* if every problem in C is polytime reducible to it.

Completeness A problem is *C-complete* if it is C-hard and it belongs to C. (The hardest problems in C)

Basic Tasks

Model-Checking Given a formula ϕ and a finite model $\mathcal{M}, w$, check whether $\mathcal{M}, w \vDash \phi$.

Satisfiability Determine satisfiability, or equivalently, validity of a formula ϕ, determine whether ϕ has a model.

M-satisfiability Determine whether ϕ is satisfiable in some model in **M**, **M** being a class of models.

Λ-**satisfiability** Determine whether ϕ is Λ-consistent, Λ being a normal modal logic. (Equivalent to $\{M^C\}$-satisfiability)

Overview

	Model-Checking	Satisfiability
Propositional logic	linear time (P)	NP
First-order logic	PSPACE	undecidable
Basic modal logic	P	PSPACE

Model Checking

Model Checking is in P in terms of the size of the model.

Let $n = |\mathcal{M}|$ denote the number of worlds in $\mathcal{M}$ and $|R|$ the number of pairs of relation R in $\mathcal{M}$. Let $\psi_1, \ldots, \psi_k$ be an enumeration of the subformulas of ϕ, in increasing length. By induction on m, it can be proved that we can mark each world in $\mathcal{M}$ with ψ_1 or $\neg\psi_i$, for $i = 1, \ldots, m$ in time $\mathcal{O}(m \cdot (|\mathcal{M}| + |R|))$. Therefore, the total complexity is $\mathcal{O}(|\phi| \cdot (|\mathcal{M}| + |R|))$.

Finite Model Property

Let Λ be a normal modal logic. Λ has the *strong finite model property* w.r.t. M if there is a *computable* function f such that every Λ-consistent formula ϕ is satisfiable in a model in M containing at most $f(|\phi|)$ states.

Λ has the *polysize model property* w.r.t. M if the function f in the above definition is polynomial.

Decidability and NP Results

Enumeration Strategy

Provided that

(a) Normal modal logic Λ has the strong finite model property w.r.t. **M**

(b) **M** is *recursive*, i.e. the membership of M is decidable.

Then Λ-satisfiability is decidable.

1. Generate all the models up to size $f(|\phi|)$.
2. Decide whether the model belongs to **M**.
3. Test for the satisfiability of ϕ on the models.

NP Results

Λ has an *NP*-complete satisfiability problem, if

1. Λ has the polysize model property w.r.t. some class of models $\mathbf{M}$.
2. The problem of deciding whether $\mathcal{M} \in \mathbf{M}$ is computable in time polynomial in $|\mathcal{M}|$.

If $\mathbf{F}$ is a class of frames definable by a first-order sentence, then the problem of deciding whether F belongs to $\mathbf{F}$ is decidable in time polynormial in the size of F. (See pp176 of the Blue Book)

Establish f.m.p.

Then how do we establish that a logic has the strong finite model property or the polysize model property?

No fully general answer to the first question is know (according to the Blue Book), but both filtration and selection can be useful.

Some of the polysize model property results boil down to a careful selection of points.

Filtration

Recall that by filtrating a model satisfying ϕ through the set of subformulas of ϕ we obtain a finite model with at most $2^{|\phi|)}$ states.

For specific logic, we need to verify that the filtrated model retains the specified properties. For example, if $\mathcal{M}$ is a **K4**-model (transitive model), then there is a filtration f such that $\mathcal{M}^f$ is transitive. (See homework 2, exercise 5) And it is possible to devise algorithm to test for transitivity of models, hence **K4** is decidable.

Theorem
$\mathbf{K}, \mathbf{T}, \mathbf{KB}, \mathbf{K4}, \mathbf{K5}, \mathbf{S4}, \mathbf{S5}, \mathbf{K_t}, \mathbf{K_t4.3}, \mathbf{K_tQ}$ *are decidable.*

Selection

Step 1, for a model $\mathcal{M}, w$ satisfying ϕ, cut $\mathit{Unr}(\mathcal{M}, w)$ to level $\deg(\phi)$. The only problem is that it can be infinitely branching.

Instead of considering all modal formulas, we only consider the subformulas of ϕ.

Step 2, starting at w, for every subformulas of ϕ of the form $\Diamond\psi$, choose a successor of w at which ψ is true, then delete all successors that were not chosen and their descendants. Now repeat this process at each of the chosen successors of the root and continue till the leaves.

Selection

Now we obtain a finite tree which satisfies ϕ at the root.

- the depth is bounded by the modal depth of ϕ.
- the branching degree is bounded by the number of diamond subformulas of ϕ.

The size of the model is exponential in the modal depth of the formulas. We will later show that the worst case is unavoidable in **K**. However, in some modal logics, by further analysis and a refined selection we can obtain a model much smaller than this.

Example: $\mathsf{K_nAlt_1}$

Theorem
$\mathbf{K_nALt_1}$, *the logic with n unary modal operators whose corresponding relations are all partial functions, has an NP-complete satisfiability problem.*

Proof.
Given a formula ϕ and a satisfying model $\mathcal{M}$, define the selection function.

$$\begin{aligned} s(p, w) &= \{w\} \\ s(\neg\phi, w) &= s(\phi, w) \\ s(\phi \wedge \psi, w) &= s(\phi, w) \cup s(\psi, w) \\ s(\Diamond_a \psi, w) &= \{w\} \cup \bigcup_{\{w' \mid R_a w w'\}} s(\psi, w') \end{aligned}$$

In the fourth clause, since every R_a is a partial function, if such a w' exists, it is unique. Therefore, $|s(\phi, w)| \leq m + 1$, where m is the number of modalities in ϕ. $\square$

Example: S4.3

Recall that **S4.3** frame is reflexive, transitive, and no right branching. We can make use of the partial order of R.

Lemma
Let F and G be two finite **S4.3** *frames, then the following two are equivalent.*

(i) *There exists a surjective bounded morphism from F to G.*
(ii) *G is isomorphic to a subframe of F that contains a maximal point of F.*

Proof.
(i) to (ii). Let f be the morphism. Let w_m be a maximal point in F, and let W' consist of w with one maximal world in $f^{-1}[v]$ for every point v of G such that $v \neq f(w)$. $F' = F|_{W'}$ is the desired subframe. $\square$

Example: S4.3

Proof.
(ii) to (i). Suppose W' contains a maximal point w_m and $F|_{W'}$ is isomorphic to G.

$$f(w) = \begin{cases} w, \text{ for } w \in W'; \\ w', f(w) \text{ is a minimal world such that } Rww' \end{cases}$$

Clearly f is surjective.

(forth) Suppose Rww'. Since $Rw'f(w')$ and R is transitive, we have $Rwf(w')$. By the minimalness of $f(w)$, $Rf(w)f(w')$.

(back) Suppose $Rf(w)f(w')$. Since $Rwf(w)$ and R is transitive, we have $Rwf(w')$. Then by the forth condition, $Rf(w)f(f(w'))$, which is $Rf(w)f(w')$. □

Example: S4.3

Theorem
The satisfiability of **S4.3** *is NP-complete.*

Proof.
Suppose ϕ is satisfiable on a finite model $\mathcal{M}$ of **S4.3**.

Let $\Diamond\psi_1, \ldots, \Diamond\psi_k$ be all the Diamond subformulas of ϕ that are satisfied at w_0. For each $1 \leq i \leq k$, select a point w_i that is maximal w.r.t. the property of satisfying ψ_i. Let w_{k+1} be a maximal point of $\mathcal{M}$. Let $\mathcal{M}'$ be the restriction of $\mathcal{M}$ on $\{w_0, w_1, \ldots, w_k, w_{k+1}\}$. $\square$

Example: S4.3

Proof.
First, $\mathcal{M}'$ is based on a **S4.3** frame. The frame underlying $\mathcal{M}'$ is a subframe that contains a maximal point of the frame of $\mathcal{M}$, and by lemma there is a surjective bounded morphism from $\mathcal{M}$ to $\mathcal{M}'$. Since such morphisms preserve modal validity, so $\mathcal{M}'$ is based on a **S4.3** frame.

Next, $\mathcal{M}', w_0 \vDash \phi$. Show that for all ϕ-subformula ψ and all i, $\mathcal{M}, w_i \vDash \psi$ iff $\mathcal{M}', w_i \vDash \psi$ by induction.

Non-trivial case: $\Diamond\psi$. Suppose $\mathcal{M}, w_i \vDash \Diamond\psi_j$. Since $\mathcal{M}$ is pointed-generated by w_0 and transitive, it follows that Rw_0w_i, hence $\mathcal{M}, w_0 \vDash \Diamond\psi_j$. Recall that w_j is a world maximal w.r.t. to the property of satisfying ψ_j, hence Rw_iw_j. By induction hypothesis, $\mathcal{M}', w_j \vDash \psi_j$, hence $\mathcal{M}', w_i \vDash \Diamond\psi$. The converse direction can be proved similarly. $\square$

Forcing Binary Trees

In **K**, we can construct a formula $A(n)$ such that the size of the smallest model satisfying $A(n)$ is exponential in $A(n)$.

$$B_i = q_i \to (\Diamond(q_{i+1} \land p_{i+1}) \land \Diamond(q_{i+1} \land \neg p_{i+1}))$$
$$S(p_i, \neg p_i) = (p_i \to \Box p_i) \land (\neg p_i \to \Box \neg p_i)$$

The B_i macro forces a branch to occur and 'splits' the truth value of p_i. The $S(p_i, \neg p_i)$ macro 'sends' the truth value assigned to p_i and its negation on e level down.

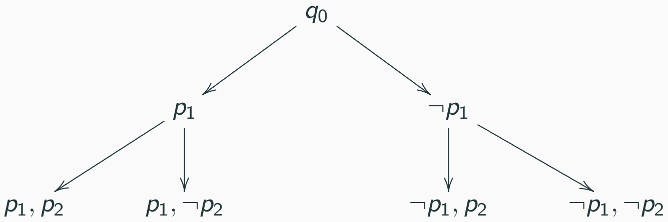
Forcing Binary Trees

(i) q_0

(ii) $\bigwedge_{1 \leq i \leq n-1} \Box^i (q_i \rightarrow \bigwedge_{i \neq j} \neg q_j)$

(iii) $B_0 \wedge \Box B_1 \wedge \cdots \wedge \Box^{n-1} B_{n-1}$

(iv) $\bigwedge_{1 \leq i \leq n-1} \Box^i \bigwedge_{0 \leq j \leq i} S(p_j, \neg p_j)$



$A(n)$ forces a binary tree in the model that satisfies it, which contains at least 2^n worlds. But the size of $A(n)$ is quadratic in n. Therefore, **K** lacks the polysize model property.

Reference

- *Modal Logic* by Patrick Blackburn et al. 2001
- *Modal Logic for Open Minds* by Johan van Benthem, 2010