The Guarded Fragment

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Our aims

Modal languages can be viewed as fragments of first order logic and these fragments have some nice computational properties(decidable and finite model property).
How far we can generalize these properties to larger fragments of first-order logic?
Define and discuss two extensions of the modal fragment with reasonably nice computational behavior.

First direction

It seems to be the fact that the modal fragment of first-order logic allows quantification only in a very restricted form:

$$ST_x(\Diamond\phi) = \exists y(Rxy \wedge ST_y(\phi))$$

First search direction: look for first-order fragments characterized by restricted quantification. It leads to the so-called guarded fragment:

$$\exists\bar{y}(G(\bar{x}, \bar{y}) \wedge \psi(\bar{x}, \bar{y}))$$

in which $G(\bar{x}, \bar{y})$ is an atomic formula and all free variables of ψ are also free in the guard $G(\bar{x}, \bar{y})$.
It can be shown that the guarded fragment has decidable satisfiability property and the finite model property.

Second direction

However, there are some very natural modal-like languages, that correspond to a decidable fragment of first-order logic as well, but are not covered by this definition. Example: temporal logic with since and until operators.

If we are interested in decidability rather than the finite model property, we could just as well settle for fragments of first-order logic to which we may apply the mosaic method.

Second search direction:try to find the so-called packed fragment of first-order logic to which the mosaic method applies, leading to a loose model property.

Some preliminaries

Language：purely relational with equality.

For a sequence of variables $\overline{x} = x_1, ..., x_n$, we write $\exists\overline{x}\phi$ as the same meaning as $\exists x_1 ... \exists x_n \phi$. Note that we view $\exists\overline{x}$ as a primitive operator.

By writing $\phi(\overline{x})$ we indicate that the free variables of ϕ are among $x_1, ..., x_n$.

Packed fragment and guarded fragment

Definition (packed fragment and guarded fragment)

A formula ϕ packs a set of variables $\{x_1, ..., x_k\}$ if (i) $Free(\phi) = \{x_1, ..., x_k\}$ and (ii) ϕ is a conjunction of the form $x_i = x_j$ or $R(x_{i_1}, ..., x_{i_n})$ or $\exists \overline{y} R(x_{i_1}, ..., x_{i_n})$ s.t. (iii) for every $x_i \neq x_j$, there is a conjunct in ϕ in which x_i and x_j both occur free.

The packed fragment *PF* is defined as the smallest set of first order formulas which contains all atomic formulas and is closed under the boolean connectives and under packed quantification. That is, whenever ψ is a packed formula, π packs $Free(\pi)$, and $Free(\psi) \subseteq Free(\pi)$, then $\exists \overline{x}(\pi \wedge \psi)$ is packed as well; π is called the guard of this formula. The guarded fragment *GF* is the subfragment of *PF* s.t. the guard π is an atomic formula.

Examples

Guarded formulas:

- the standard translation of any modal formula.
- the standard translation of any formula in the basic temporal language.
- $\forall xy(Rxy \to Ryx), \exists xy(Rxy \wedge Ryx \wedge (Rxx \vee Ryy))$.

Packed but not guarded:

- $\exists y(Rxy \wedge Py \wedge \forall z((Rxz \wedge Rzy) \to Qz))$.
- $\exists y(Rxy \wedge Py \wedge \forall z((Rxz \wedge Rzy \wedge Rxy) \to Qz))$.

Not packed(by equivalence):

$\forall yz((Rxy \wedge Ryz) \to Rxz)$.

Loose model

Let $\mathfrak{A} = (A, I)$ be a first-order structure. A tuple $(a_1, ..., a_n)$ of objects in A is called live in $\mathfrak{A}$ if either $a_1 = ... = a_n$ or $(a_1, ..., a_n) \in I(P)$ for some predicate symbol P. A subset X of A is called guarded if there is some live tuple $(a_1, ..., a_n)$ such that $X \subseteq \{a_1, ..., a_n\}$. X is packed or pairwise guarded if it is finite and each of its two-element subsets is guarded.

$\mathfrak{A}$ is a loose model of degree $k \in N$ if there is some acyclic connected graph $\mathfrak{G} = (G, E)$ and a function f mapping nodes of $\mathfrak{G}$ to subsets of A of size not exceeding k such that for every live tuple $\bar{s}$ from $\mathfrak{A}$, the set $L(\bar{s}) = \{g \in G | s_i \in f(g) \text{ for all } s_i\}$ is a non-empty and connected subset of $\mathfrak{G}$.

Main results

Theorem

Every satisfiable packed formula can be satisfied on a loose model(of degree at most the number of $\exists\bar{x}$ subformulas of ξ).

Theorem

The satisfiability problems for the guarded and the packed fragment are decidable; both problems are DEXPTIME-complete(complete for doubly exponential time). For a fixed natural number n, the satisfiability problem for formulas in the packed fragment PF_n is decidable in EXPTIME.

Theorem

Every satisfiable packed formula have a finite model.

Mosaics

Mosaics method is based on the idea of deconstructing models into a finite collection of finite submodels, and conversely, of building up new, ‘loose’, models from such parts.

Some syntactic preliminaries:

- $Var(\xi)$: the set of variables in ξ.
- $Free(\xi)$: the set of free variables in ξ .
- Let V be a set of variables. A V-substitution is any partial map $\sigma : V \rightarrow V$. Denote ψ^{σ} be the substitution σ on the formula ψ.

Definition

Let Σ be a set of packed formulas in the set V of variables. We call Σ V-closed if it is closed under subformulas, single negations and V-substitutions. With $cl_g(\xi)$ we denote the smallest $Var(\xi)$-closed set of formulas containing ξ.

Definition

Let $X \subseteq Var(\xi)$ be a set of variables. An X-type is a set $\Gamma \subseteq Cl_g(\xi)$ with free variables in X satisfying, for all formulas $\phi \wedge \psi, \neg\phi, \phi$ in $Cl_g(\xi)$ with free variables in X, the conditions (i) $\phi \wedge \psi \in \Gamma$ iff $\phi \in \Gamma$ and $\psi \in \Gamma$, (ii) $\phi \notin \Gamma$ iff $\neg\phi \in \Gamma$ and (iii) if $\phi, x_i = x_j \in \Gamma$ then $\phi^\sigma \in \Gamma$ for any substitution σ mapping x_i to x_j and/or x_j to x_i, while leaving all other variables fixed.

Mosaic

Definition

A mosaic is a pair (X, Γ) such that $X \subseteq Var(\xi)$ and $\Gamma \in Cl_g(\xi)$. A mosaic is coherent if it satisties the following conditions:

- Γ is an X-type,
- if $\psi(\bar{x}, \bar{z})$ and $\pi(\bar{x}, \bar{z})$ are in Γ, then so is $\exists \bar{y}(\psi(\bar{x}, \bar{y}) \wedge \pi(\bar{x}, \bar{y}))$.

A link between two mosaics (X, Γ) and (X', Γ') is an injective substitution σ with dom $\sigma \subseteq X$ and range $\sigma \subseteq X'$ which satisfies, for all formulas $\phi \in Cl_g(\sigma) : \phi \in \Gamma$ iff $\phi^\sigma \in \Gamma'$.

Mosaic Theorem

Definition

A requirement of a mosaic is a formula of the form $\exists \overline{y}(\psi(\overline{x},\overline{y}) \wedge \pi(\overline{x},\overline{y}))$. A mosaic (X',Γ') fulfill the requirement $\exists \overline{y}(\psi(\overline{x},\overline{y}) \wedge \pi(\overline{x},\overline{y}))$ of a mosaic (X,Γ) via the link σ if for some variables $\overline{u},\overline{v}$ in X' we have that $\sigma(\overline{x}) = \overline{u}$ and $\pi(\overline{u},\overline{v})$ and $\psi(\overline{u},\overline{v})$. a set S of mosaics is linked if every requirement of a mosaic in s is fulfilled via a link to some mosaic in S. S is a linked set of mosaics for ξ if it is linked and $\xi \in \Gamma$ for some (X,Γ) in S.

Theorem

Let ξ be a packed formula. then ξ is satisfiable if and only if there is a linked set of mosaics for ξ.

Loose models

Proof.

Left to right arrow:

We can cut out' from $\mathfrak{A}$ a linked set of mosaics for ξ. consider the set of partial assignments of elements in A to variables in $Var(\xi)$. For each α, let X_α, Γ_α be the mosaic given by $X_\alpha = dom(\alpha)$ and

$$\Gamma_\alpha = \{\phi \in Cl_g(\xi) | \mathfrak{A} \models \phi[\alpha]\}.$$

□

Right to left arrow need the following lemma:

Lemma

Let ξ be a packed formula. If there is a linked set of mosaics for ξ, then it is satisfiable in a loose model of degree $|Var(\xi)|$.