"Taming" First-Order Logic

Bounded Fragments and Others

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ML As a Proper Fragment of FOL (cf. Lecture 9)

We can translate modal formulas to first-order formulas with two variables, recording truth conditions on possible worlds models.

$$
\begin{aligned}
ST_x(p) &= Px \\
ST_x(\Diamond\phi) &= \exists y(Rxy \wedge ST_y(\phi)) \\
ST_y(\Diamond\phi) &= \exists x(Ryx \wedge ST_x(\phi))
\end{aligned}
$$

van Benthem Characterization Theorem

A first-order formula $\alpha(x)$ is equivalent to the translation of a modal formula iff it is invariant for bisimulations.

Analyzing the General Situation

Much of classical Model Theory holds for the modal fragment. The similarities call for more general explanation.

> "*Modal Logic: Bisimulation*
> *= First-Order Logic: Partial Isomorphism*"

A partial isomorphism is a one-to-one partial map between models which preserves relations both ways.

We pursue another approach, identifying key lemmas of "transfer" between modal and classical reasoning.

Upgrading Modal Equivalence to Elementary Equivalence

Lemma 1

Two models $\langle \mathcal{M}, a \rangle$ and $\langle \mathcal{N}, b \rangle$ have the same modal theory iff they possess bisimulations with two models $\langle \mathcal{M}^+, a \rangle$ and $\langle \mathcal{N}^+, b \rangle$ respectively which are elementarily equivalent.

Proof.

Upwards is immediate. Downwards: We perform *Unraveling* and *Multiplication* to yield intended models $\langle \mathcal{M}^+, a \rangle$ and $\langle \mathcal{N}^+, b \rangle$. We then use Ehrenfeucht Games to prove their elementary equivalence, employing the method of critical distance (cf. LECTURE 11) and the following sublemma. □

Sublemma 2

If the roots of two unraveled modal models have the same modal theory, then they also have the same tense-logical theory (in the basic modal language extended with a backward modal operator $\mathtt{P}$).

Proof.

Every tense-logical formula is equivalent to a Boolean combination of formulas $\mathtt{P}^i\phi$ where ϕ is purely modal.

$$\begin{aligned}
\Diamond(\mathrm{P}\alpha \wedge \beta) &\leftrightarrow \alpha \wedge \Diamond\beta \\
\Diamond(\neg\mathrm{P}\alpha \wedge \beta) &\leftrightarrow \neg\alpha \wedge \Diamond\beta \\
\mathrm{P}(\alpha \wedge \beta) &\leftrightarrow \mathrm{P}\alpha \wedge \mathrm{P}\beta \\
\mathrm{P}\neg\alpha &\leftrightarrow \neg\mathrm{P}\alpha \wedge \mathrm{P}\top
\end{aligned}$$

□

Multiplication is copying each node in each level (except the root) countably infinitely many times, performed from the root downwards: (see Blackboard)

Meta-models

The set of all first-order formulas may be viewed as *the domain of a model* with one binary relation of "semantic consequence" and another of "vocabulary inclusion". *Meta-theorems* are *first-order* statements about this model.

The modal fragemnt is a **submodel** of this meta-model.

Conjecture

The modal fragment is an elementary submodel of first-order logic.

This let one decide transfer of meta-theorems between `ML` and `FOL` by merely inspecting their syntactic form.

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Bounded Fragments

1. FRAGMENT 1 $\exists \boldsymbol{y}(R\boldsymbol{yx} \wedge \phi(\boldsymbol{y}))$
2. FRAGMENT 2 $\exists \boldsymbol{y}(R\boldsymbol{yx} \wedge \phi(\boldsymbol{x}, \boldsymbol{y}))$
3. FRAGMENT 3 $\exists \boldsymbol{y}(R\boldsymbol{yx} \wedge \phi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}))$

$R\boldsymbol{yx}$ is the *guard* of the formula. It is crucial for these fragments that guards are **atomic**.

FRAGMENT 2 is the *Guarded Fragment* that displays nice modal behaviour. A restricted version of it involves R not occurring anywhere except as a guard, with a fixed argument order $\boldsymbol{x}, \boldsymbol{y}$.

Guarded Bisimulations

A set of objects $\{a_1, \ldots, a_k\}$ is **guarded** if there exists a relation R such that $\langle a_1, \ldots, a_k \rangle \in \mathscr{I}(R)$.

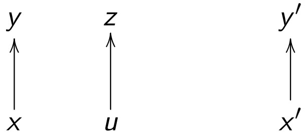
A **guarded bisimulation** is a nonempty set $\boldsymbol{F}$ of finite partial isomorphisms between two models $\mathcal{M}$ and $\mathcal{N}$ that satisfies the following Zig-Zag conditions:
Given any $f : X \to Y$ in $\boldsymbol{F}$,

1. for any Guarded $Z \subseteq M$ there is a $g \in \boldsymbol{F}$ with domain Z such that g and f agree on $X \cap Z$.
2. for any Guarded $W \subseteq N$ there is a $g \in \boldsymbol{F}$ with range W such that g^{-1} and f^{-1} agree on $Y \cap W$.

A *guarded k-bisimulation* has partial isomorphisms and mentioned guarded subsets required to be of size at most k.

An Example

$$\boldsymbol{F} = \{\{\langle x, x'\rangle\}, \{\langle z, y'\rangle\}, \{\langle x, x'\rangle, \langle y, y'\rangle\}, \{\langle u, x'\rangle, \langle z, y'\rangle\}\}.$$



Semantic Invariance

Proposition 3

Let $\boldsymbol{F}$ be a guarded bisimulation between models $\mathcal{M}$ and $\mathcal{N}$ with $f \in \boldsymbol{F}$. For all guarded formulas ϕ and all variable assignments α into the domain of f, we have $\mathcal{M}, \alpha \vDash \phi$ iff $\mathcal{N}, f \circ \alpha \vDash \phi$.

Theorem 4

A first-order formula ϕ is equivalent to a formula in $\textsc{Fragment}\ 2$ iff it is invariant for guarded bisimulations.

ϕ is equivalent to a formula in k-variable subfragment of $\textsc{Fragment}\ 2$ iff it is invariant for guarded k-bisimulations.

Unraveling Models

The Unraveling $\mathcal{M}^u$ of $\mathcal{M}$ has for its objects all pairs $\langle \pi, d \rangle$ where the path π is a finite sequence of guarded sets in M and d occurs in the final set of π but not in the one before that.

$\langle \langle \pi_i, d_i \rangle_{1 \leq i \leq k} \rangle \in \mathscr{I}(R)$ iff $\langle d_1, \ldots, d_k \rangle \in \mathscr{I}^{\mathcal{M}}(R)$ and there is a maximal path $\pi^* \in \{\pi_1, \ldots, \pi_k\}$ of which all other π_i are initial segments in such a way that their corresponding d_i remain present in each set until the end of π^*.

$\boldsymbol{F}^u$ is the family of all restrictions of the finite maps sending $\langle \pi_i, d_i \rangle$ to d_i for all guarded domains in $\mathcal{M}^u$.

Proposition 5

$\boldsymbol{F}^u$ is a guarded bisimulation from $\mathcal{M}^u$ to $\mathcal{M}$.

Proof.

(i) We check that each $f \in \boldsymbol{F}^u$ is a partial isomorphism. Preservation of relations from $\mathcal{M}^u$ to $\mathcal{M}$ is obvious. Now since $\mathrm{Dom}(f) = \{\langle \pi_i, d_i \rangle_{1 \leq i \leq k}\}$ is guarded, $\{\pi_1, \ldots, \pi_k\}$ *forms a* **chain of sequences** *and their corresponding d_i remain present in each set until the end of the π^*.* Therefore we have preservation of relations from $\mathcal{M}$ to $\mathcal{M}^u$. And if $\langle \pi, d \rangle, \langle \pi', d \rangle \in \mathrm{Dom}(f)$, then since d does not occur in the penultimate set of π and π', we have $\pi = \pi'$. Therefore we have injectivity of f. □

Proof.

Cont'd. (ii) We check the Zig-Zag conditions. Zig from $\mathcal{M}^u$ to $\mathcal{M}$ is obvious. Now consider again f above and some guarded Z in $\mathcal{M}$. Since $\{\pi_1, \ldots, \pi_k\}$ is a **chain of sequences**, there is a maximal path π^+ among objects mapped by f onto $\mathrm{Ran}(f) \cap Z$. We use π^+ or extend it by Z in case Z contain objects that are not in the last set of π^+. This latter path, matched with objects in $Z \setminus \mathrm{Ran}(f)$, induces, together with f, the desired partial isomorphism. Therefore the Zag clause from $\mathcal{M}$ to $\mathcal{M}^u$ is satisfied. □

Further Variations

Parametrized unraveling $\mathcal{M}^u(Y)$ is defiend just like $\mathcal{M}^u$ except that its paths all start from Y.
$\boldsymbol{F}^u(Y)$ is a guarded bisimulation from $\mathcal{M}^u(Y)$ to $\mathcal{M}$.

k-unraveling $\mathcal{M}^u_k$ is the submodel of $\mathcal{M}^u$ whose paths contain only sets of size at most k.
$\boldsymbol{F}^u_k$ is a guarded k-bisimulation from $\mathcal{M}^u_k$ to $\mathcal{M}$.

One can also restrict the *length*, rather than the "width" of paths in unraveled models. This is connected with restricted quantifier or modal operator depth.

Some Łoś-Tarski Theorem

Characterization of formulas which are preserved under submodels are a recurring semantic test for being a "nice fragment" of FOL.

Theorem 6

A modal formula is preserved under model extensions iff it can be defined using only propositional atoms and their negations, $\wedge$, $\vee$, $\Diamond$ **(existential form)**.

Theorem 7

A formula in the *Guarded Fragment* is preserved under submodels iff it can be defined using only atomic formulas and their negations, $\wedge$, $\vee$, $\forall$ **(universal form)**.

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Two Approaches Towards "Taming"

"Taming" first-order logic means localizing what may be called a well-behaved decidable "core-part".

"Syntactical" Approach: using standard semantics over non-standard language fragments, such as the FRAGMENTS.

"Semantical" Approach: using non-standard generalized semantics over the full standard first-order language.

This provides us with a uniform perspective on the literature.

Two Non-Standard Semantics

Modal first-order models: $\mathcal{M} = \langle S, \{R_x\}_{x \in \mathrm{VAR}}, I \rangle$ where I gives a truth value to all atomic formulas in each state $\alpha \in S$.

$$\mathcal{M}, \alpha \vDash Px \iff I(\alpha, Px)$$
$$\mathcal{M}, \alpha \vDash \exists x \phi \iff \text{for some } \beta : R_x \alpha \beta \text{ and } \mathcal{M}, \beta \vDash \phi$$

Generalized assignment models: S is a family of variable assignments $S \subseteq D^{\mathrm{VAR}}$ and R_x is the standard $=_x$ relation (identity up to x-values).

$$\mathcal{M}, \alpha \vDash \exists x \phi \iff \text{for some } \beta : \alpha =_x \beta \text{ and } \mathcal{M}, \beta \vDash \phi$$

Possible assignment gaps model "dependencies" between variables.