5.2 Algebraizing Modal Logic: the algebraic languages

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Structure of the presentation

The goal of section 5.2 is to algebraizing modal logic and present Lindenbaum-Tarski Algebras, which is analogous to the canonical models in modal logic.

Algebraizing propositional logic (Section 5.1)

Truth Value Algebras
Formulas Algebras
Homomorphism (validity preservation)

Algebraizing modal logic (Section 5.2)

Boolean Algebras with Operators (half jobs done)

Relation between Algebra and Logic

Algebra viewed as logic

Equational logic that is strongly sound and complete with respect to the standard algebraic semantics.

Logic viewed as Algebra

Advantage 1: powerful new techniques
Advantage 2: the algebraic semantics is better-behaved than framed-based semantics.

A simple example of algebra

Example

$$x^2 + 4x + 3 = 15$$

Syntax

We have rules on how to use the terms and how to manipulate them. For instance we could substitute terms that are equal.

Semantics

We use x to represent numbers and $+$ to represent operation. Different algebras have different rules of operations. For instance in matrix $xy = yx$ is not valid.

We think of propositional variables p, q, r as symbols denoting propositions, connectives $\bot, \neg, \wedge$ as denoting operations on propositions.

Algebraizing propositional semantics

Definition 5.1 (Algebras of type Bool)

Let Bool be the algebraic similarity type $\mathcal{F} = (F, \rho)$ s.t. $F = \{\bot, \neg, \vee\}$.
Given a set of propositional variables Φ, $Form(\Phi)$ is the set of Bool-terms in Φ.
Algebras of type Bool are usually presented as 4-tuples $\mathfrak{A} = (A, +, -, 0)$

For algebraizing propositional logic, however, we still need a way to present valuation of propositional logic.

Algebraizing propositional semantics

Definition 5.2 (Algebras of truth value)

$\mathbf{2} = (\{0, 1\}, +, -, 0)$

$-$ is defined by $-a = 1 - a$, and $a + b = max(a, b)$

Definition 5.3 (Algebras of formulas)

$\mathfrak{Form}(\Phi) = (\text{Form}(\Phi), +, -, \bot)$

FormΦ is the collection of propositional letters over Φ,

$-$ is defined by $-\phi = \neg\phi$, and $\phi + \psi = \phi \vee \psi$

With these algebraic correspondeces of formula and truth value, we could give the meaning of the formulas through a homomorphism.

Algebraizing propositional semantics

Valuation and Assignment

An assignment on **2** is a function $\tilde{\theta} : Form(\Phi) \to \{0, 1\}$, which is derived from the valuation θ of the propositional logic.

$$\begin{aligned} \tilde{\theta}(p) &= \theta(p), \text{ for all } p \in \Phi \\ \tilde{\theta}(\bot) &= 0 \\ \tilde{\theta}(\neg\phi) &= 1 - \tilde{\theta}(\phi) \\ \tilde{\theta}(\phi \vee \psi) &= \max(\tilde{\theta}(\phi), \tilde{\theta}(\psi)) \end{aligned} \tag{1}$$

Algebraizing propositional semantics

Proposition 5.4 (Meaning as homomorphism)

Given any assignment $\theta : \Phi \to 2$, the function $\tilde{\theta} : Form(\Phi) \to 2$ assigning to each formula its meaning under this valuation, is a homomorphism from $\mathfrak{Form}(\Phi)$ to $\mathbf{2}$.

Definition B.3 (Homomorphisms)

Let $\mathfrak{A} = (A, f_{\mathfrak{A}})_{f \in F}$ and $\mathfrak{B} = (B, f_{\mathfrak{B}})_{f \in F}$ be two algebras of the same similarity type.
A map $\eta : A \to B$ is a homomorphism if for all $f \in F$, and all $a_1, \ldots, a_n \in A$ (where n is the rank of f):

$$\eta\left(f_{\mathfrak{A}}\left(a_1, \ldots, a_n\right)\right) = f_{\mathfrak{B}}\left(\eta a_1, \ldots, \eta a_n\right)$$

The idea of viewing formulas as terms, and meaning as a homomorphism, is fundamental to algebraic logic.

Algebraizing propositional semantics

From truth to validity

An equation $s \approx t$ is valid in an algebra $\mathfrak{A}$ if for every assignment to the variables occurring in ther the terms, s and t have the same meaning in $\mathfrak{A}$.

$$\tilde{\theta}(\phi \leftrightarrow \psi) = \begin{cases} 1 & \text{if } \tilde{\theta}(\phi) = \tilde{\theta}(\psi) \\ 0 & \text{otherwise} \end{cases}$$

Theorem 5.5 (**2** Algebraizes Classical Validity)

$$\vDash_C \phi \text{ iff } \mathbf{2} \vDash \phi \approx \top \quad (5.2)$$

$$\mathbf{2} \vDash \phi \approx \psi \text{ iff } \vDash_C \phi \leftrightarrow \psi \quad (5.3)$$

$$\vDash_C \phi \leftrightarrow (\phi \leftrightarrow \top) \quad (5.4)$$

A proper, 'full' algebraization of a logic needs a translation to run forth and back.

Set Algebra

As we will see, set algebras provide us with a second algebraic perspective on the semantics of propositional logic, which will provide extends neatly to modal logic.

Let us check that set algebra does characterize validity of propositional logic. First the formulas.

Definition 5.7 (Set Algebra)

The power set algebra $\mathfrak{P}(A)$ is the structure
$$\mathfrak{P}(A) = (\mathcal{P}(A), \cup, -, \emptyset) \quad .$$
The functions are defined in the standard way, the special element $A(\emptyset)$is the top(bottom) set of the algebra.

A set algebra is a subalgebra of a power set algebra, which contains $\emptyset$ and A, and is closed under $\cup$ $-$ and $\cap$.
The class of all set algebras is called **Set** .

Set Algebra

Proposition 5.8

Every power set algebra is isomorphic to a power of **2**, and conversely.

$$\chi : \mathcal{P}(A) \to 2^A \text{ s.t. } \chi(X)(a) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha : 2^I \to \mathcal{P}(I) \text{ s.t. } \alpha(f) = \{i \in I | f(i) = 1\}.$$

Set Algebra

Theorem 5.9 (Set algebraizes classical validity)

$$\vDash_C \phi \iff \mathbf{Set} \vDash \phi \approx \top$$

Proof.

Since in 5.5 we showed $\vDash_C \phi$ iff $\mathbf{2} \vDash \phi \approx \top$, now we show that $\mathbf{Set} \vDash \phi \approx \top \iff \mathbf{2} \vDash \phi \approx \top$.

$$\begin{aligned}
\Longrightarrow: \mathbf{Set} \vDash \phi \approx \top &\Longrightarrow \mathfrak{P}(A) \vDash \phi \approx \top && \text{take any } \mathfrak{P}(A) \in \mathbf{Set} \\
&\Longrightarrow \mathbf{2}^A \vDash \phi \approx \top && \text{5.8, isomorphism holds equations} \\
&\Longrightarrow \mathbf{2} \vDash \phi \approx \top && \text{Power holds equations} \\
\Longleftarrow: \mathbf{2} \vDash \phi \approx \top &\Longrightarrow \mathbf{2}^A \vDash \phi \approx \top && \text{Power holds equations} \\
&\Longrightarrow \mathfrak{P}(A) \vDash \phi \approx \top && \text{5.8, isomorphism holds equations} \\
&\Longrightarrow \mathbf{Set} \vDash \phi \approx \top && A \text{ is arbitrary}
\end{aligned}$$

□

Algebraizing Modal Logic

We extend the Boolean algebras' language to incorporate modal operators.

Definition 5.18($\mathrm{Ter}_\tau(\Phi)$)

Let τ be a modal similarity type. The corresponding algebraic similarity type F_τ contains as function symbols all modal operators, together with the boolean symbols $\vee$ (binary), $\neg$ (unary), and $\bot$ (constant).

$$\mathrm{Form}(\tau, \Phi) = \mathrm{Ter}_\tau(\Phi)$$

Definition 5.19 (Boolean Algebras with Operators)

Let $\tau = (O, \rho)$ be a modal similarity type.
A BAOs with τ-operators is an algebra

$$\mathfrak{A} = (A, +, -, 0, f_\Delta)_{\Delta \in \tau}$$

in which f_Δ is an operator of arity $\rho(\Delta)$ that satisfies normality and additivity.

Algebraizing Modal Logic

f_Δ's normality and additivity

Normality $f_\Delta\left(a_1, \ldots, a_{\rho(\Delta)}\right) = 0$ whenever $a_i = 0$ for some $i(0 < i \leq \rho(\Delta))$

Additivity for all i (such that $0 < i \leq \rho(\Delta)$)

$$f_\Delta\left(a_1, \ldots, a_i + a_i', \ldots, a_{\rho(\Delta)}\right) = \\ f_\Delta\left(a_1, \ldots, a_i, \ldots, a_{\rho(\Delta)}\right) + f_\Delta\left(a_1, \ldots, a_i', \ldots, a_{\rho(\Delta)}\right)$$

f_Δ's monotonicity

An operation g on a boolean algebra is monotonic if $a \leq b$ implies $ga \leq gb$.
(Definition 5.10 : $a \leq b$ iff $a \cdot b = a$ iff $a + b = b$.)

Normality and additivity are two "modal" properties of f_Δ. To see this, consider a unary operator f:

$$f(0) = 0$$
$$f(x + y) = fx + fy$$

The corresponding modal formulas would be:

$$\diamond \bot \leftrightarrow \bot$$
$$\diamond(p \vee q) \leftrightarrow \diamond p \vee \diamond q$$

Monotonicity also has a modal analog:

$$\text{if } \vdash_A p \to q \text{ then } \vdash_A \diamond p \to \diamond q$$

Now we extend the Boolean algebras with regard to the modal semantics.

Definition 5.21 (Complex Algebras)

Let τ be a modal similarity type, and $\mathfrak{F} = (W, R_\Delta)_{\Delta\in\tau}$ a τ -frame. The (full) complex algebra of $\mathfrak{F}$ (notation: $\mathfrak{F}^+$), is the expansion of the power set algebra $\mathfrak{P}(W)$ with operations $m_{R_\triangle}$ for every operator Δ in τ.

$$m_R(X) = \{y \in W | \text{ there is an } x \in X \text{ such that } Ryx\}$$
$$\tilde{V}\left(\Delta\left(\phi_1, \ldots, \phi_n\right)\right) = m_{R_\Delta}\left(\tilde{V}\left(\phi_1\right), \ldots, \tilde{V}\left(\phi_n\right)\right)$$

A complex algebra is a subalgebra of a full complex algebra. If **K** is a class of frames, then we denote the class of full complex algebras of frames in **K** by **CmK**

Proposition 5.22(Complex algebra is a BAOs)

Let τ be a modal similarity type, and $\mathfrak{F} = (W, R_\Delta)_{\triangle \in \tau}$ a τ frame. Then $\mathfrak{F}^+$ is a boolean algebra with τ -operators.

(Spoiler!) Every abstract boolean algebra with operators has a concrete set theoretic representation, for every boolean algebra with operators is isomorphic to a complex algebra.

Definition 5.23 (Valuation and Assignment for BAOs)

Assume that τ is a modal similarity type and that Φ is a set of variables. Assume further that $\mathfrak{A} = (A, +, -, 0, f_\Delta)_{\Delta \in \tau}$ is a boolean algebra with τ -operators. An assignment for Φ is a function $\theta : \Phi \to A$. We can extend θ uniquely to a meaning function $\tilde{\theta} : \mathrm{Ter}_\tau(\Phi) \to A$ satisfying:

$$\begin{aligned} \tilde{\theta}(p) &= \theta(p), \text{ for all } p \in \Phi \\ \tilde{\theta}(\bot) &= 0 \\ \tilde{\theta}(\neg s) &= -\tilde{\theta}(s) \\ \tilde{\theta}(s \vee t) &= \tilde{\theta}(s) + \tilde{\theta}(t) \\ \tilde{\theta}\left(\Delta\left(s_1, \ldots, s_n\right)\right) &= f_\Delta\left(\tilde{\theta}\left(s_1\right), \ldots, \tilde{\theta}\left(s_n\right)\right) \end{aligned}$$

We say that $s \approx t$ is true in $\mathfrak{A}$ (notation: $\mathfrak{A} \models s \approx t$) if for every assignment $\theta : \tilde{\theta}(s) = \tilde{\theta}(t)$

Proposition 5.24 (Algebraizing Modal Validity)

Proposition 5.24 Let τ be a modal similarity type, ϕ a τ-formula, $\mathfrak{F}$ a τ-frame, θ an assignment (or valuation) and w a point in $\mathfrak{F}$. Then

$$(\mathfrak{F}, \theta), w \Vdash \phi \text{ iff } w \in \tilde{\theta}(\phi) \quad (5.12)$$

$$\mathfrak{F} \Vdash \phi \text{ iff } \mathfrak{F}^+ \models \phi \approx \top \quad (5.13)$$

$$\mathfrak{F}^+ \models \phi \approx \psi \text{ iff } \mathfrak{F} \Vdash \phi \leftrightarrow \psi \quad (5.14)$$

Theorem 5.25

Let τ be a modal similarity type, ϕ and ψ τ-formulas, and **K** a class of τ -frames. Then

$$\mathbf{K} \Vdash \phi \text{ iff } \mathbf{CmK} \models \phi \approx \top \quad (5.16)$$

$$\mathbf{CmK} \models \phi \approx \psi \text{ iff } \mathbf{K} \Vdash \phi \leftrightarrow \psi \quad (5.17)$$

Proof of (5.12)

$$(\mathfrak{F}, \theta), w \Vdash \phi \text{ iff } w \in \tilde{\theta}(\phi)$$

Assume that ϕ is of the form $\diamond\psi$. The key observation is that

$$\tilde{\theta}(\diamond\psi) = m_{R_\diamond}(\tilde{\theta}(\psi))$$

We now have:

$$\begin{aligned} \mathfrak{F}, \theta, w \Vdash \diamond\psi & \text{ iff there is a v s.t.} R_\diamond wv \text{ and } (\mathfrak{F}, \theta), v \Vdash \psi \\ & \text{iff there is a v s.t.} R_\diamond wv \text{ and } v \in \tilde{\theta}(\psi) \\ & \text{iff } w \in m_{R_\diamond}(\tilde{\theta}(\psi)) \\ & \text{iff } w \in \tilde{\theta}(\diamond\psi) \end{aligned}$$