Algebraizing Modal Logic

Algebraizing modal axiomatics

Wang Haoyu

Advanced modal logic

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$$\begin{array}{ccc}
\vDash_C \phi & \xleftrightarrow{\textit{iff}\ (5.9)} & \mathbf{Set} \vDash \phi \approx \top \\
\textit{iff} \updownarrow & & \updownarrow \textit{iff}\ (5.16) \\
\vdash_C \phi & \xleftrightarrow{\textit{iff}\ (5.11)} & \mathbf{BA} \vDash \phi \approx \top
\end{array}$$

Main results of 5.2

Algebraizing modal logic

- Theorem 5.25 CmK algebraizes frame validity.
- Theorem 5.27 V_Σ algebraizes modal theoremhood.
- Chapter 5.43 The Jónsson-Tarski theorem Any BAO is embeddable in the full complex algebra of its ultrafilter frame.

Let K be a class of frames and Σ a set of formulas.

$$\begin{array}{ccc}
K \models \phi & \overset{\textit{iff}\ (5.25)}{\longleftrightarrow} & \mathbf{Cm}K \models \phi \approx \top \\
?\updownarrow & & \updownarrow (5.43) \\
\vdash_{\mathbf{K}_\tau \Sigma} \phi & \overset{\textit{iff}\ (5.27)}{\longleftrightarrow} & \mathbf{V}_\Sigma \models \phi \approx \top
\end{array}$$

Algebraizing modal axiomatics

Review (c.f. B.12)

BAOs semantics

An assignment for a set of variables X w.r.t. an algebra (A, I) is a function $\theta : X \to A$. We can extend it to a meaning function $\tilde{\theta}$: Form$(X) \to A$ satisfying:

$$\tilde{\theta}(p) = \theta(p) \text{ for all } p \in X \qquad \tilde{\theta}(\bot) = 0$$

$$\tilde{\theta}(\phi_1 \vee \phi_2) = \tilde{\theta}(\phi_1) + \tilde{\theta}(\phi_2) \qquad \tilde{\theta}(\neg\phi) = -\tilde{\theta}(\phi)$$

$$\tilde{\theta}(\nabla(\phi_1, \ldots, \phi_n)) = f_\nabla(\tilde{\theta}(\phi_1) \ldots, \tilde{\theta}(\phi_n))$$

Another version of Normality and Additivity (c.f. Definition 5.19)

Recall that $1 = -0$ and $x \cdot y = -(-x + -y)$.

- Norm': $f_\nabla(a_1, \ldots, a_n) = 1$ whenever $a_i = 1$ for some $i \in [1, n]$
- Add': $f_\nabla(a_1, \ldots, a_i \cdot a'_i, \ldots, a_n)$
  $= f_\nabla(a_1, \ldots, a_i, \ldots, a_n) \cdot f_\nabla(a_1, \ldots, a'_i, \ldots, a_n)$

Theorem 5.27 (Algebraic Completeness)

- Let τ be a modal similarity type and Σ a set a of τ-formulas.
- Let $\phi^{\approx} = \phi \approx \top$. Let $\Sigma^{\approx} = \{\sigma^{\approx} | \sigma \in \Sigma\}$.
- Let $\mathbf{V}_{\Sigma}$ be the class of BAOs such that $\mathbf{V}_{\Sigma} \models \Sigma^{\approx}$.
- $\mathbf{K}_{\tau}\Sigma$ is the normal modal τ logic axiomatized by Σ.
- $\mathbf{K}_{\tau}\Sigma$ is sound and weakly complete with respect to $\mathbf{V}_{\Sigma}$,
- i.e. $\vdash_{\mathbf{K}_{\tau}\Sigma} \phi \iff \mathbf{V}_{\Sigma} \models \phi^{\approx}$ for all formulas ϕ.

Proof. $\Longrightarrow$: Suppose $\vdash_{\mathbf{K}_{\tau}\Sigma} \phi$. We show that $\mathbf{V}_{\Sigma} \models \phi^{\approx}$ by induction on the length n of the proof of ϕ in $\mathbf{K}_{\tau}\Sigma$.

$n = 1$

If ϕ is an axiom, i.e. $\phi \in \Sigma$, since $\phi^{\approx} \in \Sigma^{\approx}$, $\mathbf{V}_{\Sigma} \models \phi^{\approx}$ holds by definition.

Cont. soundness $n = 1$

- Propositional axioms:

$$
\begin{aligned}
\tilde{\theta}(p \to (q \to p)) &= \tilde{\theta}(\neg p \vee (\neg q \vee p) \\
&= (-\theta(p)) + ((-\theta(q)) + \theta(p) \\
&= ((-\theta(p)) + \theta(p)) + (-\theta(q)) \qquad (By (B0)(B1)) \\
&= 1 + (-\theta(q)) \qquad (By (B0)) \\
&= 1 (?c.f. Definition\ 5.10) = \tilde{\theta}(\top)
\end{aligned}
$$

The other two axioms can be proved similarly.

- K^i (version 1): By Add', we have $\mathbf{V}_\Sigma \models (\nabla(r_1, \ldots, r_{i-1}, p \wedge q, \ldots, r_n) \leftrightarrow (\nabla(r_1, \ldots, r_{i-1}, p \wedge q, \ldots, r_n) \wedge \nabla(r_1, \ldots, r_{i-1}, p \wedge q, \ldots, r_n)))^{\approx}$.

Cont. soundness $n = 1$

- K^i (version 2):

$$\tilde{\theta}(\nabla(r_1, \dots, r_{i-1}, p \to q, \dots, r_n) \to \nabla(r_1, \dots, r_{i-1}, p, \dots, r_n) \to \nabla(r_1, \dots, r_{i-1}, q, \dots, r_n))$$

$$= (-\tilde{\theta}(\nabla(\dots, p \to q, \dots))) + (-\tilde{\theta}(\nabla(\dots, p, \dots))) + \tilde{\theta}(\nabla(\dots, q, \dots))$$

$$= (-(f_\nabla(\dots, \tilde{\theta}(p \to q), \dots) \cdot (f_\nabla(\dots, \theta(p), \dots)))) + f_\nabla(\dots, \theta(q), \dots))$$

$$= (-f_\nabla(\dots, ((-\theta(p)) + \theta(q)) \cdot \theta(p), \dots))) + f_\nabla(\dots, \theta(q), \dots))$$

$$= (-f_\nabla(\dots, (-\theta(p)) \cdot \theta(p) + \theta(q) \cdot \theta(p), \dots))) + f_\nabla(\dots, \theta(q), \dots))$$

$$= (-f_\nabla(\dots, \theta(q) \cdot \theta(p), \dots))) + f_\nabla(\dots, \theta(q), \dots))$$

$$(?) = 1 = \tilde{\theta}(\top).$$

cont. Soundness $n > 1$ (all $\mathsf{K}_\tau\Sigma$-rules are valid on $\mathbf{V}_\Sigma$)

- MP: Suppse ϕ follows by MP from ψ and $\psi \to \phi$. By IH, $\mathbf{V}_\Sigma \vDash \psi^{\approx}$ and $\mathbf{V}_\Sigma \vDash (\psi \to \phi)^{\approx}$. That is, given any assignment θ, $\tilde{\theta}(\psi) = \tilde{\theta}(\neg\psi \vee \phi) = 1$. Therefore $\tilde{\theta}(\neg\psi) + \tilde{\theta}(\phi) = 1$. Since $\tilde{\theta}(\neg\psi) = -1 = 0$, we have $\tilde{\theta}(\phi) = 1 = \tilde{\theta}(\top)$, i.e. $\mathbf{V}_\Sigma \vDash \phi^{\approx}$.
- USUB: Suppse $\phi = \psi(p\backslash\pi)$ follows from ψ by USUB. By IH, $\mathbf{V}_\Sigma \vDash \psi^{\approx}$. Given any assignment θ, we define θ' such that $\theta'(p) = \tilde{\theta}(\pi)$. Therefore $\tilde{\theta}(\phi) = \tilde{\theta}'(\psi) = 1 = \tilde{\theta}(\top)$, i.e. $\mathbf{V}_\Sigma \vDash \phi^{\approx}$.
- NEC: Suppse $\phi = \nabla(\bot, \ldots, \psi, \ldots, \bot)$ follows from ψ by NEC. By IH, $\mathbf{V}_\Sigma \vDash \psi^{\approx}$. Thus for every R such that $Rww_1 \ldots w_n$, there is a w_i on which $\tilde{\theta}(\psi) = 1$. By Norm', $\tilde{\theta}(\phi) = \tilde{\theta}(\top)$, i.e. $\mathbf{V}_\Sigma \vDash \phi^{\approx}$. □

Corollary: $\mathbf{V}_{\mathsf{K}_\tau\Sigma} = \mathbf{V}_\Sigma$

Proof: For any $\phi \in \mathsf{K}_\tau\Sigma$, $\mathbf{V}_\Sigma \vDash \phi^{\approx}$. Therefore, $\mathbf{V}_\Sigma \subseteq \mathbf{V}_{\mathsf{K}_\tau\Sigma}$. Since $\Sigma \subseteq \mathsf{K}_\tau\Sigma$, $\mathbf{V}_{\mathsf{K}_\tau\Sigma} \subseteq \mathbf{V}_\Sigma$. Thus $\mathbf{V}_{\mathsf{K}_\tau\Sigma} = \mathbf{V}_\Sigma$. □

Towards completeness

For any ϕ, suppose $\nvdash_{\mathbf{K}_\tau\Sigma} \phi$, we need to find an algebra $\mathfrak{A}$ such that $\mathfrak{A} \in \mathbf{V}_\Sigma(\star)$ and $\mathfrak{A} \not\models \phi^{\approx}(\star\star)$. ($\mathfrak{A}$: Lindenbaum-Tarski algebra)

Let τ be an algebraic similarity type, Φ a set of propositional variables and Λ a mormal modal τ-logic.

Definition 5.28 (Formula algebra of τ over Φ)

- $\mathfrak{Form}(\tau, \Phi) = (Form(\tau, \Phi), +, -, \bot, f_\nabla)_{\nabla \in \tau}$
- $-\phi := \neg\phi$, $\phi + \psi := \phi \vee \psi$, $f_\nabla(t_1, \ldots, t_n) := \nabla(t_1, \ldots, t_n)$.

Definition 5.29

$\phi \equiv_\Lambda \psi$ iff $\vdash_\Lambda \phi \leftrightarrow \psi$ iff ϕ and ψ are equivalent modulo Λ.

Definition (Congruence)

Let $\mathfrak{A}$ be an algebra. An equivalence relation R on $\mathfrak{A}$ is a congruence iff for all $f \in \tau$, if $Ra_1b_1, \ldots, Ra_nb_n$, then $Rf_\mathfrak{A}(a_1, \ldots, a_n)f_\mathfrak{A}(b_1, \ldots, b_n)$.

A congruence relation

Proposition 5.30

- $\equiv_\Lambda$ is a congruence relation on $\mathfrak{Form}(\tau, \Phi)$.

Proof. Since $\leftrightarrow$ is an equivalence relation, $\equiv_\Lambda$ is also an equivalence relation. For the three operations in $\mathfrak{Form}(\tau, \Phi)$,

- if $\phi_i \equiv_\Lambda \psi_i$ for $i \in \{0, 1\}$, then $\vdash_\Lambda \phi_i \leftrightarrow \psi_i$ for $i \in \{0, 1\}$. By USUB, $\vdash_\Lambda \phi_0 \vee \psi_0 \leftrightarrow \phi_1 \vee \psi_1$, which implies $\phi_0 \vee \psi_0 \equiv_\Lambda \phi_1 \vee \psi_1$;
- if $\phi \equiv_\Lambda \psi$, then $\vdash_\Lambda \neg\phi\neg \leftrightarrow \psi$, followed by $\neg\phi \equiv_\Lambda \neg\psi$;
- if $\phi_i \equiv_\Lambda \psi_i$ for $i \in [1, n]$, then $\vdash_\Lambda \phi_i \leftrightarrow \psi_i$ for $i \in [1, n]$. If we can show that $\vdash_\Lambda \nabla(\phi_1, \ldots, \phi_n) \to \nabla(\psi_1, \ldots, \psi_n)$, by the symmetry between ϕ_i and ψ_i, we would have the desired result. □

A proof of $\vdash_\wedge \nabla(\phi_1, \ldots, \phi_n) \to \nabla(\psi_1, \ldots, \psi_n)$

(???)

$\phi_i \leftrightarrow \psi_i$	*Assum.*	(1)
$\bot \to \phi_i \leftrightarrow \psi_i$	P theorem	(2)
$\phi_i \to \psi_i$	$(\wedge - (1))$	(3)
$\nabla(\phi_1 \leftrightarrow \psi_1, \bot, \ldots)$	*NEC*(1)	(4)
$\nabla(\phi_1 \to \psi_1, \bot, \ldots)$	*NEC*(3)	(5)
$\nabla(\phi_1 \to \psi_1, \bot, \ldots) \to \nabla(\phi_1, \bot, \ldots) \to \nabla(\psi_1, \bot, \ldots)$	*K*	(6)
$\nabla(\phi_1, \bot, \ldots) \to \nabla(\psi_1, \bot, \ldots)$	*MP*(5)(6)	(7)
		(8)

An ideal candidate

Corollary.

Let $[\phi] = \{\psi | \phi \equiv_\Lambda \psi\}$. The following functions are well-defined.

- $[\phi] + [\psi] := [\phi \vee \psi]$
- $-[\psi] := [\neg\psi]$
- $f_\nabla([\phi_1], \ldots, [\phi_n]) := [\nabla(\phi_1, \ldots, \phi_n)]$

Definition 5.31

The **Lindenbaum-Tarski algebra** of a normal modal τ-logic Λ over the set of generators, i.e. a set of propositional variables Φ is

$$\mathfrak{L}_\Lambda(\Phi) := (Form(\tau, \Phi)/\equiv_\Lambda, +, -, f_\nabla).$$

Property ($\star\star$)

Theorem 5.32 (v.s. Theorem 5.14)
Let τ be a modal similarity type, and Λ a normal modal τ-logic. Let ϕ be some propositional formula, and Φ a set of proposition letters of size not smaller than the number of proposition letters occurring in ϕ. Then

$$\vdash_\Lambda \phi \iff \mathfrak{L}_\Lambda(\Phi) \models \phi^{\approx}.$$

Proof. Assume that Φ contains all variables occuring in ϕ.

$\Longleftarrow$: Suppose $\not\vdash_\Lambda \phi$. Then by MP, $\not\vdash_\Lambda \top \to \phi$. Then $\not\vdash_\Lambda \top \leftrightarrow \phi$, i.e. $\phi \not\equiv_\Lambda \top$ or $[\phi] \neq [\top]$. Then we define an assignment θ s.t. $\theta(p) = [p]$ for all $p \in \Phi$. We can show by induction on ϕ that $\tilde{\theta}(\phi) = [\phi]$. So $\tilde{\theta}(\phi) \neq \tilde{\theta}(\top)$. Thus $\mathfrak{L}_\Lambda(\Phi) \not\models \phi^{\approx}$.

cont. Property ($\star\star$) Soundness

$\Longrightarrow$: Let θ be an assignment s.t. $\theta(p) = [\phi_p]$ for all $p \in \Phi$. Let $\rho(\psi) = \psi(p_1 \backslash \phi_{p_1}) \dots (p_n \backslash \phi_{p_n})$ where $\{p_i | i \in [1, n]\}$ is the set of all variables occuring in ψ.

Lemma. $\tilde{\theta}(\psi) = [\rho(\psi)]$

Proof. We show it by induction on ψ.

- If $\psi = p$, then $\tilde{\theta}(\psi) = \theta(p) = [\phi_p] = [\psi(p \backslash \phi_p)] = [\rho(\psi)]$.
- If $\psi = \neg\phi$, then $\tilde{\theta}(\psi) = -\tilde{\theta}(\phi) = -[\rho(\phi)] = [\neg\rho(\phi)] = [\rho(\psi)]$.
- If $\psi = \phi_1 \vee \phi_2$, the proof is similar.
- If $\psi = \nabla(\phi_1, \dots, \phi_n)$, then $\tilde{\theta}(\psi) = f_\nabla(\tilde{\theta}(\phi_1), \dots, \tilde{\theta}(\phi_n)) = f_\nabla([\rho(\phi_1)], \dots, [\rho(\phi_n)]) = [\nabla([\rho(\phi_1)], \dots, [\rho(\phi_n)]] = [\rho(\psi)]$. $\square$

By USUB, $\vdash_\Lambda \rho(\psi)$. Therefore we have $\rho(\psi) \equiv_\Lambda \top$, i.e. $[\rho(\psi)] = [\top]$. By the lemma, we have $\tilde{\theta}(\psi) = [\top]$, i.e. $\mathfrak{L}_\Lambda(\Phi) \vDash \phi^{\approx}$. $\square$

Property $(\star)$

Theorem 5.33

Let τ be a modal similarity type, and Λ a normal modal τ-logic. Then for any set Φ of propositional letters, $\mathfrak{L}_\Lambda(\Phi) \in \mathbf{V}_\Lambda$.

Proof. With 5.32, we only have to show that $\mathfrak{L}_\Lambda(\Phi)$ is a BAO. Clearly it is a boolean algebra. We only have to show that f_∇ is indeed an operator by verifying the Add' and Norm' properties.

- Add': Since we have $\vdash_\Lambda \nabla(\phi_1, \ldots, \phi_i \wedge \phi_i', \ldots, \phi_n) \leftrightarrow \nabla(\phi_1, \ldots, \phi_i, \ldots, \phi_n) \wedge \nabla(\phi_1, \ldots, \phi_i', \ldots, \phi_n)$.
  $f_\nabla([\phi_1], \ldots, [\phi_i] \cdot [\phi_i'], \ldots, [\phi_n]) = f_\nabla([\phi_1], \ldots, [\phi_i \wedge \phi_i'], \ldots, [\phi_n])$

$=[\mathrm{f}_\nabla(\phi_1, \ldots, \phi_i \wedge \phi_i', \ldots, \phi_n)]$
$=[\mathrm{f}_\nabla(\phi_1, \ldots, \phi_i, \ldots, \phi_n) \wedge f_\nabla(\phi_1, \ldots, \phi_i', \ldots, \phi_n)]$
$=[\mathrm{f}_\nabla(\phi_1, \ldots, \phi_i, \ldots, \phi_n)] \cdot [f_\nabla(\phi_1, \ldots, \phi_i', \ldots, \phi_n)]$
$=\mathrm{f}_\nabla([\phi_1], \ldots, [\phi_i], \ldots, [\phi_n]) \cdot f_\nabla([\phi_1], \ldots, [\phi_i'], \ldots, [\phi_n])$.

cont. Property ($\star$) Normality

- Norm': Suppose there is a $a_i \in Form(\tau, \Phi)/\equiv_\Lambda$ such that $a_i = 1$, i.e. $a_i = [\top]$. Then $f_\nabla([\phi_1], \ldots, [\top], \ldots, [\phi_n]) = [f_\nabla(\phi_1, \ldots, \top, \ldots, \phi_n)] = [\nabla(\phi_1, \ldots, \top, \ldots, \phi_n)] = [\top] = 1.$ $\square$

In contrast to frame semantics

- Immediately we have: (Normal ?) modal logics are always complete w.r.t. the variety (c.f. Definition B.7) of BAOs where their axioms are valid.
- Note that modal logics are not necessarily complete w.r.t. the class of frames that they define.

Limits and further results

However...

We want completeness w.r.t. complex algebras rather than abstract BAOs.

Jónsson-Tarski theorem

Every BAO is isomorphic to a complex algebra.

By taking the complex algebra of the ultrafilter frame of a BAO, we obtain the canonical embedding algebra of the original BAO.

The filter of an algebra

Definition 5.34 (Filter of algebra v.s. filter over set)

A filter of a boolean algebra $\mathfrak{A} = (A, +, -, 0)$ is a subset $F \subseteq A$ satisfying

(F1) $1 \in F$,

(F2) If $a, b \in F$ then $a \cdot b \in F$,

(F3) If $a \in F$ and $a \leq b$ then $b \in F$.

A filter is proper if it does not contain the smallest element 0, or, equivalently, if $F = A$. An ultrafilter is a proper filter satisfying

(F4) For every $a \in A$, either a or $-$a belongs to F.

Proposition 5.38 (Ultrafilter theorem)

Let $\mathfrak{A}$ be a boolean algebra, a an element of A, and F a proper filter of A that does not contain a. Then there is an ultrafilter extending F that does not contain a.

Theorem 5.16

The Stone representation theorem

Any boolean algebra is isomorphic to a field of sets, and hence, to a subalgebra of a power of 2. As a consequence, the variety of boolean algebras is generated by the algebra 2:

$$\mathsf{BF} = \mathbb{V}(\{2\})$$

Outline of proof:

- Let $\mathfrak{A}$ be a boolean algebra and the representation function $r : A \to \mathcal{P}(Uf\mathfrak{A})$ be

$$r(a) = \{u \in Uf\mathfrak{A} | a \in u\}$$

- r is a homomorphism.
- r is injective. (by Proposition 5.38)

Definition 5.40

- The ultrafilter frame of $\mathfrak{A}$: $\mathfrak{A}_+ = (Uf\mathfrak{A}, Q_{f_\nabla})_{\nabla\in\tau}$.
- The (canonical) embedding algebra of $\mathfrak{A}$: $\mathfrak{Em}\mathfrak{A} = (\mathfrak{A}_+)^+$

Theorem 5.43 (The Jónsson-Tarski theorem)

Let $\mathfrak{A}$ be a BAO. Then the representation function $r : A \to \mathcal{P}(Uf\mathfrak{A})$ given by

$$r(a) = \{u \in Uf\mathfrak{A} | a \in u\}$$

is an embedding of $\mathfrak{A}$ into $\mathfrak{Em}\mathfrak{A}$.

Canonicity

$$
\begin{array}{lcr}
K \models \phi & \overset{\textit{iff}\ (5.25)}{\longleftrightarrow} & \mathbf{Cm}K \models \phi \approx \top \\
?\updownarrow & & \updownarrow (5.43) \\
\vdash_{\mathbf{K}_\tau \Sigma} \phi & \overset{\textit{iff}\ (5.27)}{\longleftrightarrow} & \mathbf{V}_\Sigma \models \phi \approx \top
\end{array}
$$

Exercise 5.2.6 (The complete variety of BAOs)
A variety $\mathbf{V}$ is complete if there is a frame class K that generates it, i.e. $\mathbf{V} = HSP\mathbf{Cm}K$. A logic Λ is complete iff $\mathbf{V}_\Lambda$ is a complete variety.