

Advanced Modal Logic XXIV

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1 Finitary Method: Completeness of PDL

Propositional Dynamic Logic

$$\begin{aligned}\phi &::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid [\pi]\phi \\ \pi &::= a \mid \phi? \mid (\pi; \pi) \mid (\pi + \pi) \mid \pi^*\end{aligned}$$

where $p \in \mathbf{P}$, $a \in \mathbf{\Sigma}$. The semantics is defined on Kripke models $\langle W, \{R_a \mid a \in \mathbf{\Sigma}\}, V \rangle$ with relations of *atomic actions* only. The semantics is defined as:

$$\mathcal{M}, s \vDash [\pi]\phi \iff \text{for all } t \text{ such that } sR_\pi t : \mathcal{M}, t \vDash \phi$$

where R_π is computed from those R_a where a is atomic:

$$\begin{aligned}R_{\pi_1; \pi_2} &= \{(s, t) \mid \exists u : sR_{\pi_1} u \text{ and } uR_{\pi_2} t\}; & R_{\pi_1 + \pi_2} &= R_{\pi_1} \cup R_{\pi_2}; \\ R_{\pi^*} &= \bigcup_{n>0} \underbrace{R_\pi; \dots; \pi}_n \cup \{(s, s) \mid s \in W\}, \text{ i.e., } R_{\pi^*} \text{ is the } \textit{reflexive}\end{aligned}$$

transitive closure of R_π . $sR_{\psi?}t \iff s = t \text{ and } \mathcal{M}, s \vDash \psi$.

PDL is not compact

It is clear that the following set is finitely satisfiable but it is not satisfiable as a whole:

$$\{\langle a^* \rangle p, \neg p, \neg \langle a \rangle p, \neg \langle a; a \rangle p, \neg \langle a; a; a \rangle p, \dots\}$$

Therefore we cannot expect strong completeness for sure. However, we want to prove the weak completeness which amounts to find a model for each consistent formula.

To simplify the story, we consider the test-free PDL (tfPDL) and system \mathcal{S}_{tfPDL} (\mathcal{S}_{PDL} without the TEST axiom). Note that, test-free PDL is strictly less expressive compared to the full PDL (why?).

Proof strategy

If we build a standard canonical model for S_{ffPDL} w.r.t. all the R_π^c defined by:

$$sR_\pi^c t \iff \text{for all } [\pi]\phi \in s, \phi \in t \text{ (equiv. for all } \phi \in t, \langle \pi \rangle \phi \in s)$$

then the following “truth lemma” holds trivially:

$$\mathcal{M}^c, s \Vdash \phi \iff \phi \in s$$

where $\mathcal{M}^c, s \Vdash [\pi]\phi \iff$ for all t such that $sR_\pi^c t$: $\mathcal{M}^c, t \Vdash \phi$.
 Note that R_π^c is not computed by using R_a^c where a is atomic.
 However, \vDash does not coincide with \Vdash since there may be R_{π^*} which is not the reflexive transitive closure of R_π (think about how $\{\langle a^* \rangle p, \neg p, \neg \langle a \rangle p, \neg \langle a; a \rangle p, \dots\}$ is “satisfiable” in \mathcal{M}^c).

Proof strategy

It is impossible to build a canonical model for the infinite set $\{\langle a^* \rangle p, \neg p, \neg \langle a \rangle p, \neg \langle a; a \rangle p, \dots\}$ but we can build a canonical model for each finite consistent set of formulas, in particular for a consistent formula ϕ .

Here we recall the filtration technique: we can compress a big model into a smaller one preserving the truth for a subformula-closed finite set of formulas.

Proof strategy

The strategy consists of the following steps:

- 1 Given a consistent formula, find a suitable closure of it;
- 2 According to the closure, build a filtration \mathcal{N} from \mathcal{M}^c ;
- 3 Prove that $\mathcal{N}, [s] \models \psi \iff \mathcal{M}^c, s \Vdash \psi$ for all the ψ in the closure.

Note that, we build the filtration w.r.t. R_a^c in \mathcal{M}^c , we discard the information about R_π^c for non-atomic π . Therefore we cannot use the filtration theorem directly, and need to show the filtration really preserves the truth value of the relevant formulas in (3).

Step one: closure

The Fischer-Ladner closure (FL) is defined as follows:

$$FL(p) = \{p\}$$

$$FL(\neg\phi) = \{\neg\phi\} \cup FL(\phi)$$

$$FL(\phi \wedge \psi) = \{\phi \wedge \psi\} \cup FL(\phi) \cup FL(\psi)$$

$$FL([\pi]\psi) = FL^\square([\pi]\psi) \cup FL(\psi)$$

$$FL^\square([a]\psi) = \{[a]\psi\}$$

$$FL^\square([\pi_1; \pi_2]\psi) = \{[\pi_1; \pi_2]\psi\} \cup FL^\square([\pi_1][\pi_2]\psi) \cup FL^\square([\pi_2]\psi)$$

$$FL^\square([\pi_1 + \pi_2]\psi) = \{[\pi_1 + \pi_2]\psi\} \cup FL^\square([\pi_1]\psi) \cup FL^\square([\pi_2]\psi)$$

$$FL^\square([\pi^*]\psi) = \{[\pi^*]\psi\} \cup FL^\square([\pi][\pi^*]\psi)$$

Note that FL^\square will not unravel $[\pi^*]\psi$ forever: it stops after unravelling the first $[\pi]$ in $[\pi][\pi^*]\psi$.

The closure is due to the equivalent form of the $[\pi]\phi$: some subformulas are not syntactic subformula, e.g., since

$\vdash [\pi^*]p \leftrightarrow (p \wedge [\pi][\pi^*]p)$, $[\pi][\pi^*]$ is a “hidden” subformula of $[\pi^*]\phi$.

Step one: closure

It is not hard to show the following:

Proposition

- 1 If $[\pi]\psi \in FL(\phi)$, then $\psi \in FL(\phi)$
- 2 If $[\pi_1; \pi_2]\psi \in FL(\phi)$, then $[\pi_1][\pi_2]\psi \in FL(\phi)$
- 3 If $[\pi_1 + \pi_2]\psi \in FL(\phi)$, then $[\pi_1]\psi \in FL(\phi)$, $[\pi_2]\psi \in FL(\phi)$
- 4 If $[\pi^*]\psi \in FL(\phi)$, then $[\pi][\pi^*]\psi \in FL(\phi)$
- 5 $Sub(\phi) \subseteq FL(\phi)$

Moreover, $FL(\phi)$ is indeed a finite set:

Proposition

$|FL(\phi)| \leq |\phi|$ and $|FL^\square([\pi]\psi)| \leq |\pi|$.

Step two: filtration

Given a formula ϕ , we can define the equivalence relation over \mathcal{M}^c : $s \sim t$ iff for all $\psi \in FL(\phi)$: $\psi \in s \iff \psi \in t$.

Given ϕ , the filtration $\mathcal{M}_{FL(\phi)}^c$ is a tuple $\langle W, \{R_a \mid a \in \Sigma\}, V \rangle$ where:

- $W = \{[s]_{\sim} \mid s \in \mathcal{M}^c\}$
- $uR_a v$ iff there exists $s \in u$ and $t \in v$ such that $sR_a^c t$ in \mathcal{M}^c .
- $p \in V(u)$ iff $p \in V(s)$ for all $s \in u$.

Think about the definition of V .

Step two: filtration

To show:

$$\mathcal{M}_{FL(\phi)}^c, [s] \vDash \phi \iff \mathcal{M}^c, s \Vdash \phi$$

The most crucial step is to prove:

$$\mathcal{M}_{FL(\phi)}^c, [s] \vDash [\pi]\psi \iff \mathcal{M}^c, s \Vdash [\pi]\psi$$

for non-atomic π . For that we can show the two properties of the filtration (for any π):

- If $sR_{\pi}^c t$ then $[s]R_{\pi}[t]$;
- If $[s]R_{\pi}[t]$ and $s \Vdash [\pi]\psi \in FL(\phi)$ then $t \Vdash \psi$.

Step three: $sR_{\pi}^c t$ then $[s]R_{\pi}[t]$

Induction on the structure of π : $\pi = a$ is trivial by definition.

$\pi = \pi_1; \pi_2$: if we can find a state u in \mathcal{M}^c such that: $sR_{\pi_1}^c uR_{\pi_2}^c t$, then by induction, we have $[s]R_{\pi_1}[u]R_{\pi_2}[t]$ thus $[s]R_{\pi_1; \pi_2}[t]$. This amounts to showing that the following set is consistent:

$\{\chi \mid [\pi_1]\chi \in s\} \cup \{\langle \pi_2 \rangle \psi \mid \psi \in t\}$. Suppose not, then there are a few $[\pi_1]\chi_i \in s$ and some $\psi_j \in t$ such that $\vdash \bigwedge \chi_i \rightarrow \neg \bigwedge \langle \pi_2 \rangle \psi_j$. By DISK and NEC we have

$$\vdash [\pi_1] \bigwedge \chi_i \rightarrow [\pi_1] \neg \bigwedge \langle \pi_2 \rangle \psi_j.$$

Thus $[\pi_1] \neg \bigwedge \langle \pi_2 \rangle \psi_j \in s$. Since $\bigwedge \psi_j \in t$ we have

$\langle \pi_1; \pi_2 \rangle \bigwedge \psi_j \in s$ (by definition of $R_{\pi_1; \pi_2}^c$) which contradicts to $[\pi_1] \neg \bigwedge \langle \pi_2 \rangle \psi_j \in s$.

The case of $\pi_1 + \pi_2$ is relatively easy.

Step three: $sR_{\pi}^C t$ then $[s]R_{\pi}[t]$

For π^* , the strategy for $\pi_1; \pi_2$ does not work: we may not be able find a R_{π} path to a $R_{\pi^*}^C$ -reachable state (think about a^*).

We make use of the finiteness of the filtration as follows:

- 1 use a formula to characterize R_{π^*} -reachable states from $[s]$
- 2 show that if $sR_{\pi^*}^C t$ then t satisfies that formula, thus $[t]$ is reachable from $[s]$.

Step three: $sR_{\pi}^c t$ then $[s]R_{\pi}[t]$

Now the formula is

$$\theta_{[s]} = \bigvee_{[s]R_{\pi^*}[t]} \chi_{[t]} \text{ where } \chi_{[t]} = \bigwedge_{\psi \in FL(\phi) \cap t} \psi \wedge \bigwedge_{\psi \in FL(\phi) \text{ but } \psi \notin t} \neg \psi$$

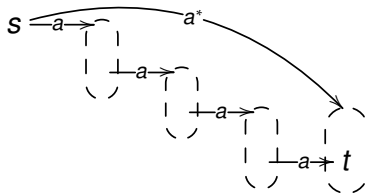
It is not hard to see that $\mathcal{M}^c, u \Vdash \theta_{[s]} \iff [s]R_{\pi^*}[u]$ (\star).

Suppose $sR_{\pi}^c t$ we need to show $t \Vdash \theta_{[s]}$

- for all t_1, t_2 , $t_1 \Vdash \theta_{[s]}$ and $[t_1]R_{\pi}[t_2]$ implies $[s]R_{\pi^*}[t_2]$ (by \star , R_{π^*})
- \Rightarrow for all t_1, t_2 , $t_1 \Vdash \theta_{[s]}$ and $[t_1]R_{\pi}[t_2]$ implies $t_2 \Vdash \theta_{[s]}$ (by \star)
- $\Rightarrow \mathcal{M}^c \Vdash \theta_{[s]} \rightarrow [\pi]\theta_{[s]}$ (by IH for R_{π}^c and the semantics)
- $\Rightarrow \mathcal{M}^c, s \Vdash \theta_{[s]} \wedge [\pi^*](\theta_{[s]} \rightarrow [\pi]\theta_{[s]})$
(by the semantics and the fact that $[s]R_{\pi^*}[s]$ and (\star))
- $\Rightarrow \theta_{[s]} \wedge [\pi^*](\theta_{[s]} \rightarrow [\pi]\theta_{[s]}) \in s$ (by truth lemma for \mathcal{M}^c w.r.t. \Vdash)
- $\Rightarrow [\pi^*]\theta_{[s]} \in s$ (by IND)
- $t \Vdash \theta_{[s]}$ (by definition of $R_{\pi^*}^c$) and the truth lemma for \Vdash

Step three: $sR_{\pi}^c t$ then $[s]R_{\pi}[t]$

The a^* -path in the filtration model can be restored in \mathcal{M}^c as follows, where the circles denotes the equivalence classes w.r.t. $FL(\phi)$:



Step three: $[s]R_\pi[t] \ \& \ s \Vdash [\pi]\psi \in FL(\phi) \implies t \Vdash \psi$

Again, proof by induction on the structure of π in $[\pi]\psi$:

$\pi = a$: if $[s]R_a[t]$ then there exist $s' \in [s], t' \in [t]$ such that $s'R_a^c t'$. Now suppose $s \Vdash [a]\psi \in FL(\phi)$ then $[a]\psi \in s$ thus $[a]\psi \in s'$ therefore $\psi \in t'$ thus $\psi \in t$, i.e., $t \Vdash \psi$.

$\pi = \pi_1; \pi_2$: if $[s]R_\pi[t]$ then there is a $[v]$ such that $[s]R_{\pi_1}[v]$ and $[v]R_{\pi_2}[t]$. Now suppose $s \Vdash [\pi_1; \pi_2]\psi \in FL(\phi)$ we have $s \Vdash [\pi_1][\pi_2]\psi$ by truth lemma and the fact that s is an MCS. Since $[\pi_1][\pi_2]\psi \in FL(\phi)$ and $[s]R_{\pi_1}[v]$, by IH, $v \Vdash [\pi_2]\psi \in FL(\phi)$. By IH again $t \Vdash \psi$.

We omit the case of $\pi_1 + \pi_2$.

Step three: $[s]R_{\pi}[t] \ \& \ s \Vdash [\pi]\psi \in FL(\phi) \implies t \Vdash \psi$

π^* : Suppose $[s]R_{\pi^*}[t]$ then there is a R_{π} -path from $[s]$ to $[t]$ via $[s_1] \dots [s_k]$. Now suppose $s \Vdash [\pi^*]\psi \in FL(\phi)$ then by FIX and the truth lemma for \mathcal{M}^c w.r.t. \Vdash , $s \Vdash [\pi][\pi^*]\psi \in FL(\phi)$, thus by IH (note that we are doing induction on the outermost $[\pi]$) $s_1 \Vdash [\pi^*]\psi$. Again, $s_1 \Vdash [\pi][\pi^*]\psi$ by FIX and the truth lemma. By IH again $s_2 \Vdash [\pi^*]\psi$. We can continue this process and obtain $t \Vdash [\pi^*]\psi$, and thus $t \Vdash \psi$ by axiom FIX and the truth lemma again.

Check the textbook for a direct construction of the finite canonical model.