Advanced Modal Logic XV

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Advanced Modal Logic (2020 Spring)

Towards the Sahlqvist fragment

Which modal formulas have first-order correspondents?
Answer 1: if ϕ has a local correspondent $\alpha(x)$ (i.e., for all $\mathcal{F}, w$: $\mathcal{F}, w \vDash \phi \iff \mathcal{F} \Vdash \alpha(x)[w]$) then it has a (global) correspondent. For example, $\Box p \to p$ has a local correspondent Rxx and it has the first-order correspondent $\forall x Rxx$.
Answer 2: if ϕ is closed (it does not mention any proposition letter) then its second-order translation is without any unary predicate thus it is a first-order formula.
Answer 3: if ϕ has some nice property then we can eliminate the universal quantifiers over valuations by considering the "representatives" of valuations only.

Towards the Sahlqvist fragment

Definition (Positive occurrences)

An occurrence of a proposition letter p is a positive occurrence if it is in the scope of an even number of negation signs; it is a negative occurrence if it is in the scope of an odd number of negation signs. A modal formula ϕ is positive in p (negative in p) if all occurrences of p are positive (negative). A formula is called positive (negative) if it is positive (negative) in all proposition letters occurring in it.

For example, q occurs positively in $\Diamond(p \to \Box q)$ while p occurs negatively in it ($\to$ is not a primitive connective). Standard translation preserves positive/negative occurrences of proposition letters. Why?

Uniform formulas

Definition (Monotonicity)

A modal formula is *upward monotone* in p if its truth is preserved under extensions of $V(p)$. Likewise, a formula is *downward monotone* in p if its truth is preserved under shrinkings of $V(p)$.

In the other words, if V and V' *only* differ in p:
ϕ is upward monotone if for any $\mathcal{F}$ and any V and V' over $\mathcal{F}$:

$$\textit{Ceteris paribus}, V(p) \subseteq V'(p) \implies [\![\phi]\!]^{\mathcal{F},V} \subseteq [\![\phi]\!]^{\mathcal{F},V'}$$

ϕ is downward monotone if for any $\mathcal{F}$ and V and any V' over $\mathcal{F}$:

$$\textit{Ceteris paribus}, V'(p) \subseteq V(p) \implies [\![\phi]\!]^{\mathcal{F},V} \subseteq [\![\phi]\!]^{\mathcal{F},V'}$$

Uniform formulas

It is not hard to show that:

Proposition

For any modal formula ϕ:
(1) If ϕ is positive in p, then it is upward monotone in p,
(2) if ϕ is negative in p, then it is downward monotone in p.

Proof.

We need a *simultaneous induction* here: prove (1) and (2) together. Recall the result about the modal preservation of ultrafilter extensions: the stronger claim is easier to prove since you can use stronger induction hypothesis. □

Uniform formulas

Definition

A modal formula is *uniform* in p if all the occurrences of p are positive or all the occurrences of p are negative. A formula is uniform if it is uniform in all proposition letters.

Theorem

Every uniform modal formula has a first-order correspondent.

Proof.

(1) Eliminating all the $\forall P$ in the (local) second-order standard translation and (2) Replacing Px by $\neg x = x$ if ϕ is positive in p and replacing Px by $x = x$ if ϕ is negative in p. (3) Make it global. □

$P(\cdot)$ can be defined by λ expressions (e.g., $\lambda u.u = u$).
Examples: $\Diamond\Diamond\Box p$, $\neg p \to \Diamond p$...

Towards the Sahlqvist fragment

What about formulas that are not uniform ($A \to \mathtt{POS}$)?

Example ($p \to \Diamond p$)

Let us try to eliminate the quantifiers in the *local* MSO translation: $\forall P(Px \to \exists y(xRy \wedge Py))$
Note that the consequent is positive in p, we just need to find the *minimal valuation* of P which can make the antecedent true.
Why? To see this, first note that for all $\mathcal{F}, w$: $\mathcal{F} \Vdash \forall P(Px \to \exists y(xRy \wedge Py))[w] \implies \mathcal{F} \Vdash (Px \to \exists y(xRy \wedge Py))[X, w]$ where X is the minimal valuation for P which makes Px true. For the converse, suppose $\mathcal{F} \Vdash Px \to \exists y(xRy \wedge Py)[X, w]$. Now consider an arbitrary $Y \subseteq W_{\mathcal{F}}$, if $\mathcal{F} \Vdash Px[Y, w]$ then $X \subseteq Y$ (since X is minimal valuation making Px true). (to be continued.)

Towards the Sahlqvist fragment

Example ($p \to \Diamond p$ cont.)

Since the consequent is positive in p thus by the upward monotonicity $\mathcal{F} \Vdash \exists y(xRy \wedge Py)[Y, w]$, thus $\mathcal{F} \Vdash (Px \to \exists y(xRy \wedge Py)[Y, w]$ for arbitrary Y. Therefore $\mathcal{F} \Vdash \forall P(Px \to \exists y(xRy \wedge Py)[w]$.
Let $Pu :=$ **u=x**, we now eliminate $\forall P$ and the antecedent: $\exists y(xRy \wedge$ **y=x**$)$ which is (locally) equivalent to xRx.

Example ($\Box p \to \Box\Box p$: antecedent with $\Box$)

$\forall P(\forall y(xRy \to Py) \to \forall y(xRy \to \forall z(yRz \to Pz)))$
Let $Pu :=$ **xRu**, we now eliminate $\forall P$ and the antecedent: $\forall y(xRy \to \forall z(yRz \to$ **xRz**$))$.

Towards the Sahlqvist fragment

Example ($\Diamond p \to \Diamond\Diamond p$: antecedent with $\Diamond$)

$\forall P(\exists y(xRy \wedge Py) \to \exists y(xRy \wedge \exists z(yRz \wedge Pz)))$
Now what is the minimal valuation satisfying the antecedent?
We need to somehow pick up an *arbitrary* y and make p true there.
Note that (if y does not appear freely in β):

$$(\exists y \alpha(y)) \to \beta \quad \leftrightarrow \quad \forall y(\alpha(y) \to \beta).$$

We can then pull out the existential quantifier:
$\forall P \forall y((xRy \wedge Py) \to \exists y(Rxy \wedge \exists z(yRz \wedge Pz)))$
Let $Pu := \mathbf{u{=}y}$, we now eliminate $\forall P$:
$\forall y(xRy \to \exists \mathbf{y}(xRy \wedge \exists z(yRz \wedge \mathbf{z{=}y})))$ Ouch!

Towards the Sahlqvist fragment

Example ($\Diamond p \to \Diamond\Diamond p$: antecedent with $\Diamond$)

$\forall P(\exists y(xRy \wedge Py) \to \exists \mathbf{t}(Rx\mathbf{t} \wedge \exists z(\mathbf{t}Rz \wedge Pz)))$

Now what is the minimal valuation satisfying the antecedent? We need to somehow pick up an *arbitrary* y and make p true there. Note that (if y does not appear in β)

$$(\exists y\ \alpha(y)) \to \beta \quad \leftrightarrow \quad \forall y(\alpha(y) \to \beta).$$

We can then pull out the existential quantifier:

$\forall P \forall y((xRy \wedge Py) \to \exists \mathbf{t}(xRy \wedge \exists \mathbf{t}(\mathbf{t}Rz \wedge Pz)))$

Let $Pu := \mathbf{u{=}y}$, we now eliminate $\forall P$:

$\forall y(xRy \to \exists \mathbf{t}(xR\mathbf{t} \wedge \exists z(tRz \wedge \mathbf{z{=}y})))$

Towards the Sahlqvist fragment

Example ($(p \wedge \Diamond\Diamond p) \rightarrow \Diamond p$: antecedent with $\wedge$)

$\forall P((Px \wedge \exists y(xRy \wedge (\exists z(yRz \wedge Pz)))) \rightarrow \exists t(xRt \wedge Pt))$ By using the following tautology (y does not appear freely in α):

$$\alpha \wedge \exists y \beta(y) \leftrightarrow \exists y(\alpha \wedge \beta(y))$$

we move the existential quantifiers:
$\forall P(\exists y \exists z(Px \wedge xRy \wedge yRz \wedge Pz) \rightarrow \exists t(xRt \wedge Pt))$
Pulling out $\exists$: $\forall P \forall y \forall z((Px \wedge xRy \wedge yRz \wedge Pz) \rightarrow \exists t(xRt \wedge Pt))$
Let $Pu :=$ **u=x** $\vee$ **u=z**, we now eliminate $\forall P$:
$\forall y \forall z(xRy \wedge yRz) \rightarrow \exists t(xRt \wedge ($**t=x** $\vee$ **t=z**$)))$ which can be simplified as: $\forall y \forall z(xRy \wedge yRz) \rightarrow (xRx \vee xRz)))$.

Towards the Sahlqvist fragment

Example ($(p \vee \Box p) \to \Diamond p$: antecedent with $\vee$)

$\forall P((Px \vee \forall y(xRy \to Py)) \to (\exists z(xRz \wedge Pz))$ By using the following tautology:

$$(\alpha \vee \beta) \to \gamma \quad \leftrightarrow \quad ((\alpha \to \gamma) \wedge (\beta \to \gamma))$$

we turn the disjunction into a conjunction: $\forall P((Px \to (\exists z(xRz \wedge Pz))) \wedge (\forall y(xRy \to Py) \to (\exists z(xRz \wedge Pz))))$. By using the following equality:

$$\forall \ldots (\alpha \wedge \beta) \quad \leftrightarrow \quad (\forall \ldots \alpha \wedge \forall \ldots \beta)$$

$\forall P((Px \to (\exists z(xRz \wedge Pz))) \wedge \forall P(\forall y(xRy \to Py) \to (\exists z(xRz \wedge Pz)))$.
We can continue with each of the two conjuncts.

Towards the Sahlqvist fragment

Example (Geach Axiom: $\Diamond\Box p \rightarrow \Box\Diamond p$: $\Diamond$ over $\Box$)

$\forall P(\exists y(xRy \wedge \forall z(yRz \rightarrow Pz)) \rightarrow \forall s(xRs \rightarrow \exists t(sRt \wedge Pt)))$

We can then pull out the existential quantifier:

$\forall P \forall y(xRy \wedge \forall z(yRz \rightarrow Pz)) \rightarrow \forall s(xRs \rightarrow \exists t(sRt \wedge Pt)))$

Let $Pu := \mathbf{yRu}$, we now eliminate $\forall P$:

$\forall y(xRy \rightarrow \forall s(xRs \rightarrow \exists t(sRt \wedge \mathbf{yRt})))$

Towards the Sahlqvist fragment

Example ($(p \wedge \Diamond\neg p) \rightarrow \Diamond p$: antecedent with negative formulas)

$\forall P(Px \wedge \exists y(xRy \wedge \neg Py) \rightarrow \exists z(xRz \wedge Pz))$
By using the following tautology (y does not appear in α):

$$\alpha \wedge \exists y \beta(y) \leftrightarrow \exists y(\alpha \wedge \beta(y))$$

we push the existential quantifier out of conjunctions:
$\forall P(\exists y(Px \wedge xRy \wedge \neg Py) \rightarrow \exists z(xRz \wedge Pz))$
Pulling out the existential quantifier:
$\forall P \forall y((Px \wedge xRy \wedge \neg Py) \rightarrow \exists z(xRz \wedge Pz))$

Towards the Sahlqvist fragment

Example ($(p \wedge \Diamond\neg p) \rightarrow \Diamond p$: cont.)

Pulling out the existential quantifier:
$\forall P \forall y((Px \wedge xRy \wedge \neg Py) \rightarrow \exists z(xRz \wedge Pz))$
By using the following tautology:

$$(\alpha \wedge \beta) \rightarrow \gamma \quad \leftrightarrow \quad \alpha \rightarrow (\neg\beta \vee \gamma)$$

(note that we can also massage the modal formula at the first place) we move the negative part to the consequent:
$\forall P \forall y((Px \wedge xRy) \rightarrow (Py \vee \exists z(xRz \wedge Pz)))$
Let $Pu := \mathbf{u{=}x}$, we now eliminate $\forall P$:
$\forall y(xRy \rightarrow \mathbf{y{=}x} \vee (\exists z(xRz \wedge \mathbf{z{=}x})))$
It can be simplified as $\forall y(xRy \rightarrow y = x \vee xRx)$.