Advanced Modal Logic XIII

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Advanced Modal Logic (2020 Spring)

Frames and validity

Why do we study frames?

- Frames as tools for analysing modal logics (as a set of valid formulas), driven by syntactic approaches to modal logic. Can we characterize a logic by a class of frames via validity (Is system S sound and complete w.r.t. a class of frames)? How can we show $\nvdash_S \phi$? From $\forall$ to $\exists$
- Frames as structures to be described by modal logics via validity. What classes of frames are definable by modal logic? What about its *expressive power over frames* compared with classical logics?
- Important modal axioms characterize natural frame conditions. The connection gives us a better understanding of both.

Models, frames, satisfiability and validity

	local	global	local class	global class
models	$\mathcal{M}, w \models \phi$	$\mathcal{M} \models \phi$	$\mathrm{K}_{pm} \models \phi$	$\mathrm{K}_m \models \phi$
frames	$\mathcal{F}, w \models \phi$	$\mathcal{F} \models \phi$	$\mathrm{K}_{pf} \models \phi$	$\mathrm{K}_f \models \phi$

In terms of the validity over classes of pointed models:

$\mathcal{M}, w \models \phi \iff \{\mathcal{M}, w\} \models \phi$

$\mathcal{M} \models \phi \iff \{\mathcal{M}, w \mid w \in W_{\mathcal{M}}\} \models \phi$

$\mathrm{K}_m \models \phi \iff \{\mathcal{M}, w \mid \mathcal{M} \in \mathrm{K}_m\} \models \phi$

$\mathcal{F}, w \models \phi \iff \{\mathcal{M}, w \mid \mathcal{M} \text{ is based on } \mathcal{F}, w\} \models \phi$

$\mathcal{F} \models \phi \iff \{\mathcal{M}, w \mid \mathcal{M} \text{ is based on } \mathcal{F}\} \models \phi$

$\mathrm{K}_{pf} \models \phi \iff \{\mathcal{M}, w \mid \mathcal{M}, w \text{ is based on } (\mathcal{F}, w) \in \mathrm{K}_{pf}\} \models \phi$

$\mathrm{K}_f \models \phi \iff \{\mathcal{M}, w \mid \mathcal{M} \text{ is based on } \mathcal{F} \in \mathrm{K}_f\} \models \phi$

Models, frames, satisfiability and validity

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ϕ defines $\mathcal{M}, w$ modulo $\leftrightarroweq$ if for all $\mathcal{N}, v$: $\mathcal{N}, v \models \phi \iff \mathcal{M}, w \leftrightarroweq \mathcal{N}, v$
ϕ defines $\mathcal{M}$ modulo $\leftrightarroweq$ if for all $\mathcal{N}$: $\mathcal{N} \models \phi \iff \mathcal{M} \leftrightarroweq_{total} \mathcal{N}$
ϕ defines K_{pm} if for all $\mathcal{M}, w$: $\mathcal{M}, w \models \phi \iff \mathcal{M}, w \in \mathrm{K}_{pm}$
ϕ defines K_m if for all $\mathcal{M}$: $\mathcal{M} \models \phi \iff \mathcal{M} \in \mathrm{K}_m$
ϕ defines $\mathcal{F}, w$ if for all $\mathcal{F}', w'$: $\mathcal{F}', w' \models \phi \iff \mathcal{F}, w \approx_? \mathcal{F}', w'$
ϕ defines $\mathcal{F}$ if for all $\mathcal{F}'$: $\mathcal{F}' \models \phi \iff \mathcal{F}' \approx_? \mathcal{F}$
ϕ defines K_{pf} if for all $\mathcal{F}, w$: $\mathcal{F}, w \models \phi \iff \mathcal{F}, w \in \mathrm{K}_{pf}$
ϕ defines K_f if for all $\mathcal{F}$: $\mathcal{F} \models \phi \iff \mathcal{F} \in \mathrm{K}_f$

Each ϕ must define a class of models/frames, but may not define any model or frame.

Models, Frames, satisfiability and validity

ϕ can be replaced by a set Φ. Natural questions: what kind of . . . can be define by ϕ/Φ.
We can also define relative definability, for example:
ϕ defines $\mathtt{K}_f$ within C if for all $\mathcal{F} \in C$: $\mathcal{F} \models \phi \iff \mathcal{F} \in \mathtt{K}_f$. ($C$ itself may not be definable..) Q: intersection of C and $\mathtt{K}$?
Q: Can we cast the definability of classes of frames on the definability of classes of models?
ϕ defines $\mathtt{K}_f$?iff? ϕ defines $\{\mathcal{M}, w \mid \mathcal{M} \text{ is based on } \mathcal{F} \in \mathtt{K}_f\}$
$\mathtt{K}_f$ is modally definable ?iff? $\{\mathcal{M}, w \mid \mathcal{M} \text{ is based on } \mathcal{F} \in \mathtt{K}_f\}$ is modally definable

Frame Definability

Example (modal definability of classes of frames)

$p \to \Diamond p$ defines the class of reflexive frames, i.e., for all $\mathcal{F}$:
$\mathcal{F} \vDash \phi \iff \mathcal{F}$ is reflexive. ($\Box p \to p$ can also define this class)
$\Box p \to \Diamond p$ defines the class of serial frames (every world has a successor). Ex. What about $(\Box p \wedge p) \to \Diamond p$?
$\Box p \to \Box\Box p$ defines the class of transitive frames.
$p \leftrightarrow \Box p$ defines the class of frames which consist of isolated reflexive points.
$\Box\bot$ defines the class of frames which consist of isolated irreflexive points.

Q: Can two non-equivalent modal formulas define the same class of frames? Some tricky things in proving such results: to show $\forall \mathcal{F} : \mathcal{F} \notin \mathrm{K} \implies \mathcal{F} \nvDash \phi$ we need to find a counter example (model) of ϕ for *each* $\mathcal{F} \notin \mathrm{K}$.

Frame Definability

A frame can be viewed as a first-order structure for the language with equality and R_∇ but NO unary predicates P (first-order frame language).

Example (first-order definability of classes of frames)

$\forall x\, xRx$ defines the class of reflexive frames, i.e., for all $\mathcal{F}$:
$\mathcal{F} \Vdash \forall x\, xRx \iff \mathcal{F}$ is reflexive.
$\forall x \exists y\, xRy$ defines the class of serial frames.
$\forall x \forall y\, (Rxy \leftrightarrow x = y)$ defines the class of frames consisting of isolated reflexive points.
$\forall x \forall y\, \neg Rxy$ defines the class of frames consisting of isolated irreflexive points.

Frame and validity

The validity of modal formulas on frames is essentially (monadic) second-order since we need predicate variables over sets of possible worlds. The second-order frame language here is based on the first-order one with a **P**-indexed collection of monadic predicate variables which can be quantified over.

	local	global
models	$\mathcal{M} \Vdash ST_x(\phi)[w]$	$\mathcal{M} \Vdash \forall x ST_x(\phi)$
frames	$\mathcal{F} \Vdash \forall P_1 \cdots \forall P_n ST_x(\phi)[w]$	$\mathcal{F} \Vdash \forall P_1 \cdots \forall P_n \forall x ST_x(\phi)$

Frame and validity

Theorem

$$\mathcal{F}, w \vDash \phi \iff \mathcal{F} \Vdash \forall P_1 \cdots \forall P_n ST_x(\phi)[w]$$
$$\mathcal{F} \vDash \phi \iff \mathcal{F} \Vdash \forall P_1 \cdots \forall P_n \forall x ST_x(\phi)$$

Proof.

$$(\mathcal{F}, V), w \vDash \phi \iff \mathcal{F} \Vdash ST_x(\phi)[w, V(p_1) \cdots V(p_n)]$$

□

Digression: Monadic Second Order Logic (MSO)

(Büchi): A language of finite words is recognisable by a finite state automaton if and only if it is MSO definable over finite words. (How do you define the words with even number of symbols?)

(Büchi): A language of infinite words is recognisable by a Nondeterministic Büchi Automaton if and only if it is MSO definable over infinite words.

(Thatcher,Wright) A set of finite trees is recognizable by a finite tree automaton iff it is MSO definable over finite trees.

(Rabin's theorem): MSO over n-successor infinite trees (SnS) is decidable.

(Janin and Walukiewicz): Modal mu-calculus is the bisimulation invariant fragment of MSO.