Advanced Modal Logic VIII

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Advanced Modal Logic (2020 Spring)

Finite model property via filtration

Definition (A filtration of a model through Σ)

Let Σ be a subformula-closed set of modal formulas. Let $\equiv_\Sigma$ be the logical equivalence w.r.t. the formulas in Σ. A filtration of a given Kripke model $\mathcal{M}$ of a unary similarity type is the tuple $(W_{\equiv_\Sigma}, \{\xrightarrow{a}_f\}_{a \in \mathbf{O}}, V_f)$ satisfying:

- If $w \xrightarrow{a} v$ then $|w| \xrightarrow{a}_f |v|$
- If $|w| \xrightarrow{a}_f |v|$ then for all $\Diamond_a \phi \in \Sigma$, if $\mathcal{M}, v \models \phi$ then $\mathcal{M}, w \models \Diamond_a \phi$.
- $V_f(|w|) = V(w)|_{\mathbf{P}'}$ where $\mathbf{P}' \subseteq \mathbf{P}$ is the set of proposition letters in Σ.

where W_Σ is the set of equivalence classes w.r.t. $\equiv_\Sigma$ over W.

Finite model property via filtration

Proposition

If $\mathcal{M}^f$ is a filtration of $\mathcal{M}$ through a finite subformula-closed set Σ then $\mathcal{M}^f$ has at most 2^n states where $n = |\Sigma|$.

Theorem (filtration theorem)

If $\mathcal{M}^f$ is a filtration of $\mathcal{M}$ through a finite subformula-closed set Σ then for all $\phi \in \Sigma$ and w in $\mathcal{M}$ we have
$\mathcal{M}, w \vDash \phi \iff \mathcal{M}^f, |w| \vDash \phi$

Finite model property via filtration

By induction on ϕ.

The case of basic proposition is trivial. IH: for all the subformula ψ of ϕ: $\mathcal{M}, w \vDash \psi \iff \mathcal{M}^f, |w| \vDash \psi$
Boolean cases are trivial. Now consider $\Diamond_a\psi$:suppose $\mathcal{M}, w \vDash \Diamond_a\psi$ then there a v such that $w \xrightarrow{a} v$ and $\mathcal{M}, v \vDash \psi$.By definition of $\xrightarrow{a}_f$ and IH, it is clear $\mathcal{M}^f, |w| \vDash \Diamond_a\psi$. Suppose $\mathcal{M}^f, |w| \vDash \Diamond_a\psi$ then there is a $|v|$ such that $|w| \xrightarrow{a}_f |v|$ and $\mathcal{M}^f, |v| \vDash \psi$. By IH, $\mathcal{M}, v \vDash \psi$. Therefore by definition of $\xrightarrow{a}_f$, $\mathcal{M}, w \vDash \Diamond\psi$. $\square$

Q: Can we relax the first condition on $\rightarrow_f$?
Wait, but $\Diamond$ is not the *primitive* modality in our language, it can happen that $\Box\phi \in \Sigma$ but $\Diamond\phi$ $(\neg\Box\neg\phi)$ is not.

If $\Box$ is the primitive modality

Definition (A filtration of a model through Σ)

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Finite model property via filtration

Filtrations exist:

- $\xrightarrow{a}_1 = \{(|w|, |v|) \mid \text{there exist } w' \in |w|, v' \in |v| : w' \xrightarrow{a} v'\}$
- $\xrightarrow{a}_2 = \{(|w|, |v|) \mid \text{for all } \Box\phi \in \Sigma\text{: } \mathcal{M}, w \vDash \Box\phi \implies \mathcal{M}, v \vDash \phi.\}$

The second relation is well-defined: the exact choices of w, v in their equivalence classes do not matter.

Any $\xrightarrow{a}_f : \xrightarrow{a}_1 \subseteq \xrightarrow{a}_f \subseteq \xrightarrow{a}_2$ (flexible choices)

Q: Are filtrations always (restricted) bisimilar to the original model? No!

Theorem (ML has the *strong* finite model property)

If a modal logic formula ϕ is satisfiable then it is satisfiable in a finite model with the size bounded by 2^n where n is the number of subformulas of ϕ.

How many subformulas are there for a given ϕ? Less or equal than the length of the formula.

Testing Validity and Satisfiability

Entscheidungsproblem (Hilbert 1928, dates back to Leibniz): find an algorithm such that given a mathematical statement expressed by first-order logic as an input, it can output true or false correctly. Church and Turing: it is impossible!

The satisfiability problem of a semantically given logic $(L, \mathbb{C}, \vDash)$ is the problem of testing whether L formulas are satisfiable in some model in $\mathbb{C}$.

$\mathtt{ML}$ over the class of all the models has the *strong* finite model property which implies that it is decidable: just try these finite models (with finitely many propositions, modulo isomorphism) one by one (assuming the model checking problem is decidable).

What about a logic that has the (strong) finite model property in general? It still can be undecidable e.g., first-order logic on finite models, Ex 6.2.4 and Ex 6.2.5

- Strong finite model property w.r.t. recursive class of models implies decidability (you can check whether a model is in $\mathbb{C}$).
- There are decidable logics which do not have finite model property.
- Finite model property and finite axiomatization implies decidability.

Definition (Another definition of finite model property)

A logic L (a set of formulas) has f.m.p. iff every non-theorem of L has a finite counter L-model iff L is characterized by a class of finite models.

The last yet important model construction method

How to turn a model into an m-saturated one?
We need to add some successors such that every finitely satisfiable set of formulas is satisfiable in one of the successors.
How to do it?

Ultrafilter

Given a set W, a *filter* F over W is a subset of $\mathcal{P}(W)$ s.t.:

- $W \in F$
- $X, Y \in F$ implies $X \cap Y \in F$
- $X \in F$ and $X \subseteq Y$ implies $Y \in F$

A proper filter is a filter such that $\emptyset \notin F$. An *ultrafilter* is a proper filter such that either $X \in F$ or $W \backslash X \in F$.

Example

Given the set of natural numbers $\mathbb{N}$, the set

$$\{X \mid X \text{ is a co-finite subset of } \mathbb{N}\} = \{X \mid \mathbb{N} \backslash X \text{ is finite }\}$$

is a proper filter.

Ultrafilter

The first intuition: a subset of W can be viewed as (the extension) of a formula which holds exactly on the states in this subset. From this point of view, a filter is a set of formulas which is closed under $\wedge$ and $\rightarrow$. A proper filter is a consistent set and an ultrafilter is a maximal consistent set of formulas.

A *principal ultrafilter* π_w is an ultrafilter generated by a singleton set $\{w\} : \pi_w = \{X \mid w \in X \subseteq W\}$ (check that it is indeed an ultrafilter).

Then it is not hard to see that a non-principal ultrafilter (if exists) contains only infinite subsets and all the co-finite subsets of W. It also means that there is no non-principal ultrafilter over a finite W.

Ultrafilter

Theorem (Ultrafilter Theorem)

Any proper filter can be extended into an ultrafilter.

Proof.

By Zorn lemma (a version of the axiom of choice). □

Corollary

Any non-empty set E of $\mathcal{P}(W)$ can be extended into an ultrafilter iff E has the finite intersection property (any finite intersection of elements in E is non-empty).

Therefore to construct an ultrafilter from a non-empty set $E \subseteq \mathcal{P}(W)$, we just need to verify whether E has finite intersection property. To build a non-principal ultrafilter over an infinite set W, we can start from the proper filter of all the co-finite subsets of W, and apply the ultrafilter theorem.

Ultrafilter extension

Definition (Ultrafilter extension)

Given a model $\mathcal{M} = \langle W, \rightarrow, V \rangle$, its ultrafilter extension $\mathcal{M}^{\mathrm{ue}} = \langle W^{\mathrm{ue}}, \rightarrow^{\mathrm{ue}}, V^{\mathrm{ue}} \rangle$ where:

- $W^{\mathrm{ue}} = \{u \mid u \text{ is an ultrafilter over } W\}$
- $u \rightarrow^{\mathrm{ue}} u' \iff (\forall X : X \in u' \implies m_R(X) \in u)$
- $V^{\mathrm{ue}}(u) = \{p \mid \{w \mid p \in V(w)\} \in u\}$

where $m_R(X) = \{w \mid \exists v \in X \text{ such that } wRv\}$

Proposition (alternative definition of $\rightarrow^{\mathrm{ue}}$ which is more useful)

$u \rightarrow^{\mathrm{ue}} u' \iff (\forall Y : l_R(Y) \in u \implies Y \in u')$

The intuition behind two definitions of $u \rightarrow^{\mathrm{ue}} u'$:
$(\forall \phi : \phi \in u' \implies \Diamond\phi \in u)$ and $(\forall \phi : \Box\phi \in u \implies \phi \in u')$.

Ultrafilter extension

It is not hard to show that: $\pi_w \to^{\mathrm{ue}} \pi_v \iff w \to v$ (left to right: take $\{v\} \in \pi_v$).
Therefore the submodel of $\mathcal{M}^{\mathrm{ue}}$ obtained by restricting to the principal ultrafilters is an isomorphic copy of $\mathcal{M}$. The extra worlds in $\mathcal{M}^{\mathrm{ue}}$ are non-principal ultrafilters. By the ultrafilter theorem and its corollary, such non-principal ultrafilters exist if W is infinite. This justifies the name: ultrafilter **extension.**
What is the ulrafilter extension of $(\mathbb{N}, <)$?

Ultrafilter extension

Given $\mathcal{M}$, we abuse the notation of $V_{\mathcal{M}}$ and let $V_{\mathcal{M}}(\phi)$ be the set of worlds in $\mathcal{M}$ where ϕ is true.

Theorem

Given a pointed model $\mathcal{M}, w$, $\mathcal{M}, w \equiv_{ML} \mathcal{M}^{\mathrm{ue}}, \pi_w$.

It is a bit hard to prove this theorem directly since the induction hypothesis would be only about principal ultrafilters in $\mathcal{M}^{\mathrm{ue}}$, but clearly a principal ultrafilter π_w may have a successor which is a non-principal ultrafilter given that W is infinite. Thus we prove the following more general result first:

Theorem

Given a pointed model $\mathcal{M}, w$, $\mathcal{M}^{\mathrm{ue}}, u \vDash \phi \iff V_{\mathcal{M}}(\phi) \in u$.

Ultrafilter extension

by induction on ϕ.

To handle the Boolean cases, you need to use the properties of ultrafilters. They are not that trivial.
The non-trivial case of $\Box\psi$: if $\mathcal{M}^{\mathrm{ue}}, u \nvDash \Box\psi$ then there is a u' such that $u \rightarrow^{\mathrm{ue}} u'$ and $u' \nvDash \psi$. By IH, $V(\psi) \notin u'$. By the definition of $\rightarrow^{\mathrm{ue}}$, for all X $l_R(X) \in u$ implies $X \in u'$. Then $l_R(V(\psi)) \notin u$, namely $V(\Box\psi) \notin u$ by IH.
Now suppose $V(\Box\psi) \notin u$, we need to show $\mathcal{M}^{\mathrm{ue}}, u \nvDash \Box\psi$. The proof strategy is that we **construct** a successor u' of u such that $V(\psi) \notin u'$. Let $u'' = \{V(\neg\psi)\} \cup \{Y \mid l_R(Y) \in u\}$, we just need to show that it has finite intersection property. First note that $V(\neg\Box\psi) \cap l_R(Y) \neq \emptyset$ (since u is an ultrafilter and $V(\Box\psi) \notin u$ but $l_R(Y) \in u$), thus $V(\neg\psi) \cap Y$ is not empty. Moreover, based on the fact that $\{Y \mid l_R(Y) \in u\}$ is closed under intersection we can prove the theorem. □

Ultrafilter extension

Theorem

Given a pointed model $\mathcal{M}$, $\mathcal{M}^{\mathrm{ue}}$ is m-saturated.

Proof.

The idea: given a world u in $\mathcal{M}^{\mathrm{ue}}$ and a set of formulas which is finitely satisfiable in the set of successors of u, we construct a u' such that $u \to^{\mathrm{ue}} u'$ and for all $\phi \in \Sigma : V(\phi) \in u'$. Let $u'' = \{Y \mid l_R(Y) \in u\} \cup \{V(\phi) \mid \phi = \psi_0 \wedge \cdots \wedge \psi_n, n \geq 0, \psi_k \in \Sigma\}$, we need to show that u'' has the finite intersection property. Since such a ϕ is satisfiable at some successor of u thus $V(\Diamond\phi) \in u$. Similar as before we can show that $V(\phi) \cap Y \neq \emptyset$.

□

Bisimilarity somewhere else

Based on the previous results we can show:

Theorem

For any pointed models $\mathcal{M}, w$ and $\mathcal{N}, v$:
$\mathcal{M}, w \equiv_{ML} \mathcal{N}, v \implies \mathcal{M}^{\mathrm{ue}}, \pi_w \underline{\leftrightarrow} \mathcal{N}^{\mathrm{ue}}, \pi_v$.

Proof.

By a detour:

$$
\begin{array}{ccc}
\mathcal{M}, w & \equiv_{\mathtt{ML}} & \mathcal{N}, v \\
\mathtt{ML}{\equiv}\big\downarrow & & \mathtt{ML}{\equiv}\big\downarrow \\
\mathcal{M}^{\mathrm{ue}}, \pi_w & \equiv_{\mathtt{ML}} = \underline{\leftrightarrow} & \mathcal{N}^{\mathrm{ue}}, \pi_v \\
\text{m-saturated} & & \text{m-saturated}
\end{array}
$$

□