Advanced Modal Logic VII

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Advanced Modal Logic (2020 Spring)

The use of bisimilarity in understanding ML

- to measure the distinguishing power of ML. It preserves the truth values of modal formulas. We can define truth preserving operations on models based on it.
- to understand the *expressive power* of modal logic: which properties of the models can be expressed by modal formulas (modal definability)? A property is *not* definable by the basic modal logic if there are two bisimilar models such that one satisfies the property and one does not.

Model constructions

Example (truth-preserving model construction methods)

- Bounded morphism: $\mathcal{M}, w \leftrightarroweq \mathcal{N}, f(w)$
- Bisimulation contraction: $\mathcal{M}_{\leftrightarroweq}, |w| \leftrightarroweq \mathcal{M}, w$
- Generated submodel: $Gen(\mathcal{M}, X), w \leftrightarroweq \mathcal{M}, w \ (w \in X)$
- Disjoint Union: $\biguplus_i \mathcal{M}_i, w_i \leftrightarroweq \mathcal{M}_i, w_i$
- Unravelling: $Unr(\mathcal{M}, w), \langle w \rangle \leftrightarroweq \mathcal{M}, w$

Bisimulation contraction

Definition (Bisimulation contraction)

Given a Kripke model $\mathcal{M}$ of a unary similarity type, the bisimulation contraction of $\mathcal{M}$ is the quotient model $\mathcal{M}_{\underline{\leftrightarrow}} = \langle W', \{\stackrel{a}{\to}'\}_{a \in \mathbf{O}}, V' \rangle$ where:

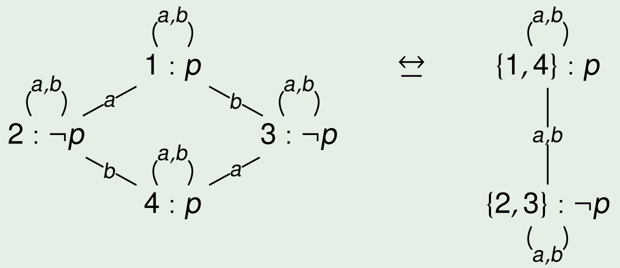
- $W' = \{|w| \mid w \in W_{\mathcal{M}}\}$ where $|w| = \{v \mid w \,\underline{\leftrightarrow}\, v\}$
- $|w| \stackrel{a}{\to}' |v|$ iff $\exists w' \in |w| \exists v' \in |v|$ such that $w' \stackrel{a}{\to} v'$ in $\mathcal{M}$
- $V'(|w|) = V(w)$

It is also fine to replace $\exists\exists$ by $\forall\exists$ in the definition of $\stackrel{a}{\to}'$.
By defining a bisimulation, we can show that

$$\mathcal{M}_{\underline{\leftrightarrow}}, |w| \,\underline{\leftrightarrow}\, \mathcal{M}, w$$

Bisimulation contraction

Example (Making models smaller!)



Therefore counting modalities is not *definable* on arbitrary models: e.g., $\mathcal{M}, w \vDash E_4\phi \iff$ there are exactly 4 worlds in $\mathcal{M}$ that satisfy ϕ. Q: Is bisimulation contraction the smallest model that is total-bisimilar to the original model?

Generated submodel

Definition (Submodel)

Given a Kripke model $\mathcal{M}$ of a unary similarity type, a submodel $\mathcal{M}'$ of $\mathcal{M}$ is a Kripke model such that $W_{\mathcal{M}'} \subseteq W_{\mathcal{M}}$ and for all $w', w'' \in W_{\mathcal{M}'}$: $w' \xrightarrow{a}_{\mathcal{M}'} w'' \iff w' \xrightarrow{a}_{\mathcal{M}} w''$ (i.e., $\xrightarrow{a}_{\mathcal{M}'} = \xrightarrow{a}_{\mathcal{M}}|_{W_{\mathcal{M}'} \times W_{\mathcal{M}'}}$), and for all $w \in W_{\mathcal{M}'}$: $V_{\mathcal{M}'}(w) = V_{\mathcal{M}}(w)$

Definition (Generated submodel)

Given a Kripke model $\mathcal{M}$ of a unary similarity type, a generated submodel $\mathcal{M}'$ of $\mathcal{M}$ is a submodel of $\mathcal{M}$ such that if $w \xrightarrow{a}_{\mathcal{M}} w'$ and $w \in W_{\mathcal{M}'}$ then $w' \in W_{\mathcal{M}'}$ (closed under transitions).

If $\mathcal{M}'$ is a generated submodel of $\mathcal{M}$ then for all $w \in W_{\mathcal{M}'}$: $\mathcal{M}', w \leftrightarroweq \mathcal{M}, w$.

Generated submodel

Definition (Generated submodel w.r.t. X)

Given a Kripke model $\mathcal{M}$ of a unary similarity type, the generated submodel $\mathcal{M}'$ of $\mathcal{M}$ w.r.t. $X \subseteq W_{\mathcal{M}}$ is the smallest submodel of $\mathcal{M}$ such that $X \subseteq W_{\mathcal{M}'}$ and if ($w \xrightarrow{a}_{\mathcal{M}} w'$ and $w \in W_{\mathcal{M}'}$) then $w' \in W_{\mathcal{M}'}$. If X is a singleton then the generated submodel is said to be point generated.

Is backward looking modality definable? $\mathcal{M}, w \models \langle a \rangle^{\leftarrow} \phi \iff$ there exists v such that $v \xrightarrow{a} w$ and $\mathcal{M}, v \models \phi$.

Disjoint Union

Two models are disjoint if they do not share a common possible world.

Definition (Disjoint Union)

Given a set of mutually disjoint Kripke models $\{\mathcal{M}_i \mid i \in I\}$ of a unary similarity type, the *disjoint union* of $\mathcal{M}_i$ is a Kripke model $\biguplus_{i \in I} \mathcal{M}_i$ such that

- $W_{\biguplus_{i \in I} \mathcal{M}_i} = \bigcup_{i \in I} W_{\mathcal{M}_i}$
- for all $\Box_a \in \mathbf{O}$: $\xrightarrow{a}_{\biguplus_{i \in I} \mathcal{M}_i} = \bigcup_{i \in I} \xrightarrow{a}_{\mathcal{M}_i}$
- $V_{\biguplus_{i \in I} \mathcal{M}_i}(w) = V_{\mathcal{M}_i}(w)$ if $w \in W_{\mathcal{M}_i}$

What if $\mathcal{M}_i$ are not disjoint? We then take isomorphic copies.
For any $w \in W_{\mathcal{M}_j}$: $\biguplus_{i \in I} \mathcal{M}_i, w \leftrightarroweq \mathcal{M}_j, w$.Definability?

(Tree) unravelling

Definition (Unravelling)

Given a pointed Kripke model $\mathcal{M}, w$ of a unary similarity type, the *unravelling* of $\mathcal{M}$ at w is a pointed Kripke model $Unr(\mathcal{M}, w), \langle w \rangle$ such that

- $W_{Unr(\mathcal{M},w)} = \{s \mid s = w_0 w_1 \cdots w_k$ such that $w_0 = w$ and for $0 \leq i < k$: $w_i \xrightarrow{a} w_{i+1}$ for some $\Box_a \in \mathbf{O}\ \}$
- for all $\Box_a \in \mathbf{O}$: $s \xrightarrow{a}_{Unr(\mathcal{M},w)} s'$ if $s = w \ldots w_k$ and $s' = s w_{k+1}$ and $w_k \xrightarrow{a}_{\mathcal{M}} w_{k+1}$
- $V_{Unr(\mathcal{M},w)}(w_0 \ldots w_k) = V_{\mathcal{M}}(w_k)$

In the above definition, we allow multiple transitions from one world to another, e.g., $s \xrightarrow{a} t$ and $s \xrightarrow{b} t$. It is fine for most applications of unravelling. To rule out such cases, we can define the set of worlds as the mixed sequences of worlds and labels, e.g., $w_0 a_0 w_1 a_1 \cdots w_k$. $Unr(\mathcal{M}, w), \langle w \rangle \leftrightarrows \mathcal{M}, w$.

Finite model property

Definition (Finite model property)

A logic $\langle L, \mathbb{C}, \vDash \rangle$ has the *finite model property* if for any ϕ in this logical language: ϕ is satisfiable at a model in $\mathbb{C}$ implies it is satisfiable at a finite model ($W_{\mathcal{M}}$ is finite) in $\mathbb{C}$.

First-order logic on the class of arbitrary models does not have finite model property:
e.g., $\forall x \neg (x < x) \wedge \forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z)$
$\wedge \forall x \exists y (x < y)$ has only infinite models.
A *set* of modal formulas may only have infinite models, e.g.,
$\{\Diamond\Box\bot, \Diamond\#\Box\bot, \Diamond\#\#\Box\bot \dots\}$ where $\#\phi := \Box\phi \wedge \Diamond\phi$.

Finite model property over arbitrary models

Theorem ($\mathtt{ML}$ has the finite model property)

If a modal logic formula is satisfiable then it is satisfiable in a finite model.

Proof.

Given a formula ϕ, suppose it is satisfiable at a pointed model $\mathcal{M}, w$. Let $k = deg(\phi)$, we cut $Unr(\mathcal{M}, w)$ up to “level” k (call the resulting model $Unr(\mathcal{M}, w)|_k$). We can show $\mathcal{M}, w \leftrightarroweq_k Unr(\mathcal{M}, w)|_k$. Then we prune the branches according to $\mathtt{ML}_k$ equivalence (over finitely many basic propositions in ϕ) such that it is finitely branching. □

Therefore $\mathtt{ML}$ cannot define classes consisting of only infinite models.