Advanced Modal Logic V

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Advanced Modal Logic (2020 Spring)

The discovery of bisimulation

- In Modal logic: van Benthem (1976) (based on the work of Segerberg (1971), de Jongh and Troelstra (1966) + the insight of a relational definition)
- In Computer Science: Park (1981) (based on the work of Milner (1980) + the insight of the greatest fixed point)
- In (non-well-founded) Set theory: Forti and Honsell (1981) Hinnion, and it was made popular by Aczel (1988)

Yet another perspective to look at bisimulaiton is through *games*.

Abstract games

A (typical turn-based) *game* consists of:

- a set of players: I
- a set of configurations (with an initial one): C
- a player assignment $f : C \to I$
- (deterministic) actions connecting configurations
- number of runs
- winning/losing conditions

A (memory-less) *strategy* of a player for a game is a function assigning to each configuration that belongs to him or her a move to proceed (if possible). We assume all the information about the game before and during the plays are known to the players (*perfect information* games), in particular the current configuration is known to the players during a play.

Determinacy of a game

A game is said to be *determined* if one of the players has a *winning strategy* (guarantees winning, no matter what others may do).

Theorem (Zemelo? Von Neumann?)

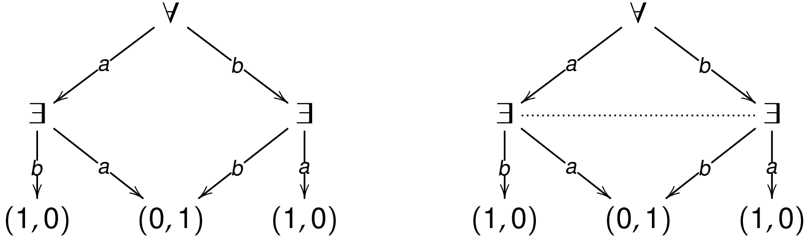
2-player finite-depth deterministic perfect-information win/lose games are determined.

Proof 1: bottom-up labelling the winning configurations.

Proof 2: the first player has a winning strategy iff $\exists a_0 \forall b_0 \exists a_1 \forall b_1 \ldots \exists a_n \forall b_n : \langle a_0 b_0 \cdots a_n b_n \rangle \in WC(1)$ then the first player does not have a winning strategy iff $\neg(\exists a_0 \forall b_0 \exists a_1 \forall b_1 \ldots \exists a_n \forall b_n : \langle a_0 b_0 \cdots a_n b_n \rangle \in WC(1))$iff $\forall a_0 \exists b_0 \forall a_1 \ldots \forall a_n \exists b_n : \langle a_0 b_0 \cdots a_n b_n \rangle \notin WC(1)$ iff Player 2 has a winning strategy. Excluded middle implies determinacy. □

Determinacy of a game

What about non-deterministic actions? What about three players? What about imperfect information games?



$$\forall x \exists y\ x = y \text{ v.s. } \forall x \exists y_{/\forall x} x = y$$

∀ and ∃ are often called Abelard and Eloise (Heloise) in game semantics (from Hodges).
Design a game semantics for modal logic (exercise).

Chess

Chess is a finite game (given the regulation on draw). Note that White always moves first. Let White-Chess be the game just like normal Chess but count a draw in the normal game as a win for white, similar for Black-Chess.

White-Chess	Black-Chess	Chess
White has a w.s.	White has a w.s.	White has a w.s
Black has a w.s.	White has a w.s.	Impossible
White has a w.s.	Black has a w.s.	Both have non-losing strategie
Black has a w.s.	Black has a w.s.	Black has a w.s.

Corollary (Zemelo 1913)

In Chess, either White can force a win, or Black can force a win, or both sides can force a draw

Bisimulation game

Definition (n-round Bisimulation Game)

An n-round bisimulation game $\mathcal{G}_n((\mathcal{M}, w), (\mathcal{N}, v))$ between $(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ is a two player game based on the configurations in $W_{\mathcal{M}} \times W_{\mathcal{N}}$. The initial configuration is (w, v) and the players, Spoiler and Defender, play in rounds. At each configuration (w', v'):

1. Spoiler selects a state w'' in $\mathcal{M}$ such that $w' \xrightarrow{a}_{\mathcal{M}} w''$ for some $a \in \mathbf{O}$ and then Defender needs to select a state v'' in $\mathcal{N}$ such that $v' \xrightarrow{a}_{\mathcal{N}} v''$. The configuration is then changed to (w'', v'').
2. Similar for the case when Spoiler first selects a state v'' in $\mathcal{N}$.

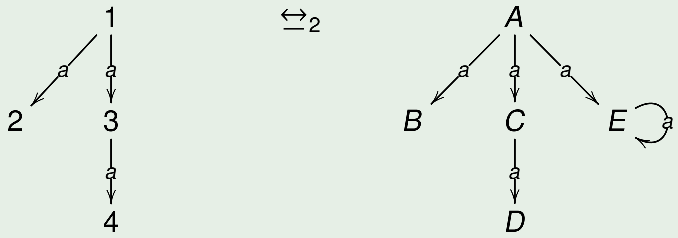
To be extremely precise we need to define configurations as $W \times W \times \{S, D\} \times \mathbf{O} \times \{1, ..., n\}$ and then the strategies will be indeed functions on configurations.

Bisimulation game

Spoiler wins the game if within n rounds some configuration (w', v') is reached such that $V_{\mathcal{M}}(w') \neq V_{\mathcal{N}}(v')$ or it is Defender's turn but she does not have a legal move to do.
Defender wins the game otherwise, i.e., either Spoiler gets stuck at some point, or because Defender can respond with legal moves for the duration of the game.
We say that Defender has a *winning strategy* in the n-round bisimulation game, if she has responses to any challenges from Spoiler that guarantee her to win the game. Winning strategies of Spoiler is defined similarly.

Bisimulation Game

Example



Defender can win k-round games ($k < 3$)

$\underline{\leftrightarrow}_0 = \{1,2,3,4\} \times \{A,B,C,D,E\}$

$\underline{\leftrightarrow}_1 =$
$\{(2,B),(2,D),(4,B),(4,D),(1,A),(1,C),(1,E),(3,A),(3,C),(3,E)\}$

$\underline{\leftrightarrow}_2 = \{(2,B),(2,D),(4,B),(4,D),(1,A),(3,C)\}$

$\underline{\leftrightarrow}_3 = \{(2,B),(2,D),(4,B),(4,D),(3,C)\}$ so $1 \underline{\not\leftrightarrow}_3 A$

Approximations of bisimilarity

Definition (n-bisimilarity $\leftrightarroweq_n$)

1. for any $\mathcal{M}, w$,$\mathcal{N}, v$: $\mathcal{M}, w \leftrightarroweq_0 \mathcal{N}, v$ iff $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$
2. $\mathcal{M}, w \leftrightarroweq_{n+1} \mathcal{N}, v$ if:
   - $\mathcal{M}, w \leftrightarroweq_n \mathcal{N}, v$
   - for any a, if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there is a v' such that $v \xrightarrow{a} v'$ and $\mathcal{M}, w' \leftrightarroweq_n \mathcal{N}, v'$
   - for any a, if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there is a w' such that $w \xrightarrow{a} w'$ and $\mathcal{M}, w' \leftrightarroweq_n \mathcal{N}, v'$

Approximations of bisimilarity

Definition (n-bisimilarity $\leftrightarroweq_n$: an equivalent definition)

1. for any $\mathcal{M}, w$,$\mathcal{N}, v$: $\mathcal{M}, w \leftrightarroweq_0 \mathcal{N}, v$ iff $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$
2. $\mathcal{M}, w \leftrightarroweq_{n+1} \mathcal{N}, v$ if:
   - $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$
   - for any a, if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there is a v' such that $v \xrightarrow{a} v'$ and $\mathcal{M}, w' \leftrightarroweq_n \mathcal{N}, v'$
   - for any a, if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there is a w' such that $w \xrightarrow{a} w'$ and $\mathcal{M}, w' \leftrightarroweq_n \mathcal{N}, v'$

Bisimulation game

Theorem (Adequacy)

For all $n \in \mathbb{N}$, $\mathcal{M}, w \leftrightarroweq_n \mathcal{N}, v$ iff Defender has a winning strategy of the n-round bisimulation game $\mathcal{G}_n((\mathcal{M}, w), (\mathcal{N}, v))$.

Proof: by induction on n

- $n = 0$: Defender has a winning strategy of the 0-round bisimulation game iff $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$ iff $\mathcal{M}, w \leftrightarroweq_0 \mathcal{N}, v$
- Inductive hypothesis (IH): for $n \leq k$ the statement holds.

to be continued...

Bisimulation game

cont.

- $n = k + 1$: $\Leftarrow$: Suppose Defender has a winning strategy in the $k+1$-round game then she has a winning strategy in the k-round game. From IH we have $\mathcal{M}, w \leftrightarroweq_k \mathcal{N}, v$. Now we check the "Zig-Zag" conditions. Suppose $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there is a v' such that $v \xrightarrow{a} v'$ and (w', v') is also a winning configuration for Defender in the next k rounds (since Defender has a winning strategy in the $k+1$-round game). By IH $w' \leftrightarroweq_k v'$. Similar for the case of $v \xrightarrow{a} v'$ in $\mathcal{N}$. Therefore $\mathcal{M}, w \leftrightarroweq_{k+1} \mathcal{N}, v$.
  $\Rightarrow$: a winning strategy for Defender starts with: " try to keep the $\leftrightarroweq_k$ pairs of worlds then keep the $\leftrightarroweq_{k-1}$ pairs ..." (Axiom of choice is needed here.)

$\square$

Bisimulation game

Theorem

$\mathcal{M}, w \underline{\leftrightarrow}_\omega \mathcal{N}, v$ iff Defender has a winning strategy in the n-round bisimulation game $\mathcal{G}_n((\mathcal{M}, w), (\mathcal{N}, v))$ for each n.

What about playing the game infinitely long?
The winning condition for Defender in the infinite game is simply: "playing forever (if Spoiler had not already lost)".

Infinite Bisimulation Game

Theorem

$\mathcal{M}, w \leftrightarroweq \mathcal{N}, v$ iff Defender has a winning strategy in the infinite bisimulation game $\mathcal{G}_\infty((\mathcal{M}, w), (\mathcal{N}, v))$.

Proof.

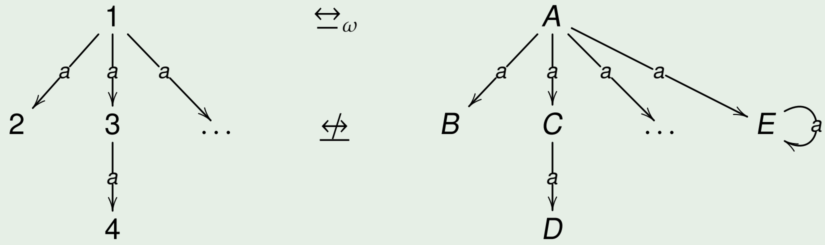
$\Rightarrow$: The winning strategy is: "Keep the bisimilar pairs". (Here you need the axiom of choice...)
$\Leftarrow$: let $Z = \{(w', v') \mid$ it is Spoiler's turn to move at (w', v') and (w', v') is a winning position for Defender $\}$. Here a winning position means that Defender can win if the game continues from this configuration. Show that Z is a bisimulation. □

Corollary

$\mathcal{M}, w \leftrightarroweq \mathcal{N}, v$ implies $\mathcal{M}, w \leftrightarroweq_\omega \mathcal{N}, v$.

Bisimulation and ω-bisimulation

Example



Can you find other examples (about $\underline{\leftrightarrow}_{\omega^2}$ and $\underline{\leftrightarrow}$)?