Basics in Modal Logic IV

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Advanced Modal Logic (2020 Spring)

1. Bisimulation

Recap

We want some structural equivalence notion to match the logical equivalence of basic modal logic.

The discovery of bisimulation in modal logic:

- Homomorphism: too weak
- Strong homomorphism: still too weak
- Surjective strong homomorphism: too strong
- Change the surjectivity to Zig and Zag: bounded morphism. Still too strong!
- Function to relation: bisimulation. Is it alright?

Bisimulation

Definition (Bisimulation)

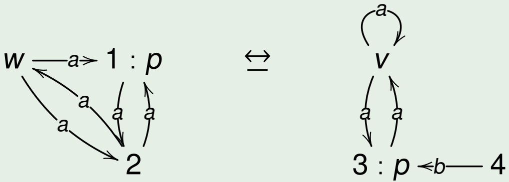
A *non-empty* binary relation Z between the domains of two models $\mathcal{M}$ and $\mathcal{N}$ is called a *bisimulation* iff whenever $(w, v) \in Z$ the following conditions are satisfied:

- **Invariance** For any $p \in \mathbf{P} : p \in V_{\mathcal{M}}(w) \iff p \in V_{\mathcal{N}}(v)$.
- **Zig** if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there exists a v' in $\mathcal{N}$ such that $v \xrightarrow{a} v'$ and $w'Zv'$.
- **Zag** if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there exists a w' in $\mathcal{M}$ such that $w \xrightarrow{a} w'$ and $w'Zv'$.

$(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ are said to be *bisimilar* $(\mathcal{M}, w \leftrightarroweq \mathcal{N}, v)$ if there is a bisimulation Z between them such that $(w, v) \in Z$. We say a bisimulation Z is *total*, if every world in one model is linked by Z to some world in the other model (notation: $\mathcal{M} \leftrightarroweq \mathcal{N}$).

Bisimulation

Example



$Z = \{(w, v), (2, v), (1, 3)\}$.

We say a bisimulation Z is *total*, if every world in one model is linked by Z to some world in the other model.

About the notations

A *bisimulation* between $\mathcal{M}, \mathcal{N}$ is a relation $Z \subseteq W_{\mathcal{M}} \times W_{\mathcal{N}}$. Is it an equivalence relation by definition?

Bisimilarity ($\underline{\leftrightarrow}$) is the *equivalence relation* between pointed models (why?) such that $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{N}, v$ iff *there is* a bisimulation between $\mathcal{M}$ and $\mathcal{N}$ linking w and v. When the two models are clear from the context we also write $w \underline{\leftrightarrow} v$ for $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{N}, v$.

However, in practice we often talk about *bisimilarity* by using the word *bisimulation*, e.g., when we say bisimulation is finer than trace equivalence we actually mean: $\underline{\leftrightarrow} \subseteq \approx_{trace}$; modal logic is invariant under bisimulation: $\underline{\leftrightarrow} \subseteq \equiv_{\mathtt{ML}}$.

Another reason for this abuse of terminology is that bisimilarity can be viewed as the union of all bisimulation (or say the largest bisimulation). We will come back to this later.

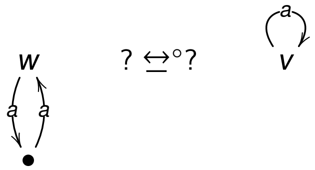
What about this definition?

$\mathcal{M}, w \underline{\leftrightarrow}^\circ \mathcal{N}, v$ **iff**:

Invariance For any $p \in \mathbf{P} : p \in V_{\mathcal{M}}(w) \iff p \in V_{\mathcal{N}}(v)$.

Zig if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there exists a v' in $\mathcal{N}$ such that $v \xrightarrow{a} v'$ and $\mathcal{M}, w' \underline{\leftrightarrow}^\circ \mathcal{N}, v'$.

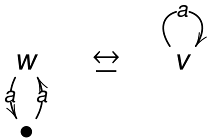
Zag if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there exists an w' in $\mathcal{M}$ such that $w \xrightarrow{a} w'$ and $\mathcal{M}, w' \underline{\leftrightarrow}^\circ \mathcal{N}, v'$.



The original definition (massaged version for defining bisimilarity between pointed models directly):

$\mathcal{M}, w_0 \leftrightarroweq \mathcal{N}, v_0$ iff there exists a non-empty $Z \subseteq W_{\mathcal{M}} \times W_{\mathcal{N}}$ such that $(w_0, v_0) \in Z$ and for any $(w, v) \in Z$

- **Invariance** For any $p \in \mathbf{P} : p \in V_{\mathcal{M}}(w) \iff p \in V_{\mathcal{N}}(v)$.
- **Zig** if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there exists a v' in $\mathcal{N}$ such that $v \xrightarrow{a} v'$ and $w'Zv'$.
- **Zag** if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there exists a w' in $\mathcal{M}$ such that $w \xrightarrow{a} w'$ and $w'Zv'$.



There is no circularity here.

Features of bisimilarity

Bisimilarity (within one model) can be viewed as the union of all the possible bisimulations: the maximal bisimulation, and it is indeed an equivalence relation.

- locality
- not an inductive definition but a *coinductive* one
- coninductive proof method

Cold jokes:
A comathmatician is a device for turning cotheorems into ffe.
Every nut is a coconut.
`https://math.stackexchange.com/questions/900390/joke-explanation-a-comathematician-is-a-device-for-turning-coth`

Induction and coinduction

Induction:

- construction from the basis
- the least solution $X^\star$ of an inequality: $f(X) \subseteq X$ (closure property)
- proof method: $f(Y) \subseteq Y \implies X^\star \subseteq Y$: $X^\star$ has property Y.

Coinduction:

- destruction from the whole
- the greatest solution $X^\star$ of an inequality: $X \subseteq f(X)$
- proof method: $Y \subseteq f(Y) \implies Y \subseteq X^\star$: Y has property $X^\star$.

Actually, coinduction can be viewed as induction on the complement, let $Y = \overline{X}$, and let $g(Y) = \overline{f(\overline{Y})}$, then $X \subseteq f(X)$ can be reformalized as $g(Y) \subseteq Y$. The least solution of Y is the greatest solution of X for $X \subseteq f(X)$.

Examples

Defining the set of points that are reachable from a point w_0 in a model as the least solution of X such that $f(X) \subseteq X$ where:

$$f(X) = \{w \mid w_0 \to w\} \cup \{w \mid \exists v \in X : v \to w\}$$

Defining the set of points that have infinite descending chains in a model as the greatest solution of X such that $X \subseteq f(X)$ where:

$$f(X) = \{w \mid \exists v \in X : w \to v\}$$

A modal μ-calculus formula: $\nu X.\Diamond X$ (where ν is the greatest fixed point operator) can thus define all the points that have infinite descending chains.

Induction and coinduction

To guarantee those least/greatest solutions do exist:

Lemma

Let $\mu = \bigcap\{X \mid f(X) \subseteq X\}$ and $\nu = \bigcup\{X \mid X \subseteq f(X)\}$. If f is a order-preserving (monotone) function ($X \subseteq Y \implies f(X) \subseteq f(Y)$) then $f(\mu) \subseteq \mu$ and $\nu \subseteq f(\nu)$.

Based on this we can show:

Theorem (Knaster-Tarski, on power sets over W)

If f is a monotone function on subsets of U: $\mathcal{P}(W) \to \mathcal{P}(W)$ then

- *μ is the least fixed point of f.*
- *ν is the greatest fixed point of f.*

Moreover, we can reach μ and ν by (transfinite) iteration of f from $\emptyset$ or W respectively (Kleene's fixed point theorem.)

Fixed-point perspective

Fix a model $\mathcal{M}$. Let us consider the function $f_{\underline{\leftrightarrow}} : \mathcal{P}(W_{\mathcal{M}} \times W_{\mathcal{M}}) \to \mathcal{P}(W_{\mathcal{M}} \times W_{\mathcal{M}})$. $f_{\underline{\leftrightarrow}}(Z)$ is defined as the set of pairs (w, v) such that:

- **Invariance** For any $p \in \mathbf{P} : p \in V_{\mathcal{M}}(w) \iff p \in V_{\mathcal{M}}(v)$.
- **Zig** if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there exists a v' in $\mathcal{M}$ such that $v \xrightarrow{a} v'$ and $w'Zv'$.
- **Zag** if $v \xrightarrow{a} v'$ in $\mathcal{M}$ then there exists an v' in $\mathcal{M}$ such that $w \xrightarrow{a} w'$ and $w'Zv'$.

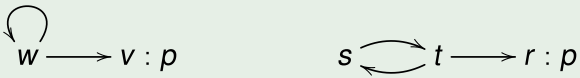
Bisimilarity within this model ($\{(w, v) \mid \mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}, v\}$) is the greatest fixed point of $f_{\underline{\leftrightarrow}}$ (the greatest solution of the inequation $Z \subseteq f_{\underline{\leftrightarrow}}(Z)$. Note that each (non-empty) solution is simply a bisimulation, the union of all the bismulations is the greatest fixed point). Existence of the fixed point is guaranteed by Knaster-Tarski fixed-point theorem ($f_{\underline{\leftrightarrow}}$ is monotone). What is the least fixed point?

To find bisimilarity in finite models

The fixed point view also tells us a way to find bisimilarity in practice (Kleene fixed point theorem):

$$Z_0 = W \times W, Z_1 = f_{\underline{\leftrightarrow}}(Z_0), Z_2 = f_{\underline{\leftrightarrow}}(Z_1), \ldots$$

Example (we only focus on the cross-model pairs)



$\{(w,s),(w,t),(w,r),(v,s),(v,t),(v,r)\}$
$\{(w,s),(w,t),(v,r)\}$
$\{(w,t),(v,r)\}$
$\{(v,r)\}$

Approximations of bisimilarity

If we start from $\{(w, v) \mid V(w) = V(v)\}$ then we will have the $\underline{\leftrightarrow}_k$ approximations of $\underline{\leftrightarrow}$.

Definition (n-bisimilarity $\underline{\leftrightarrow}_n$)

1. for any $\mathcal{M}, w, \mathcal{N}, v$: $\mathcal{M}, w \underline{\leftrightarrow}_0 \mathcal{N}, v$ iff $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$
2. $\mathcal{M}, w \underline{\leftrightarrow}_{n+1} \mathcal{N}, v$ if:
   - $\mathcal{M}, w \underline{\leftrightarrow}_n \mathcal{N}, v$
   - for any a, if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there is a v' such that $v \xrightarrow{a} v'$ and $\mathcal{M}, w' \underline{\leftrightarrow}_n \mathcal{N}, v'$
   - for any a, if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there is a w' such that $w \xrightarrow{a} w'$ and $\mathcal{M}, w' \underline{\leftrightarrow}_n \mathcal{N}, v'$

(Compare this to the definition 2.30 in the blue book). Can we replace the first condition $(\mathcal{M}, w \underline{\leftrightarrow}_n \mathcal{N}, v)$ by $V_{\mathcal{M}}(w) = V_{\mathcal{N}}(v)$? Let $\underline{\leftrightarrow}_\omega = \bigcap_{n \geq 0} \underline{\leftrightarrow}_n$ we **will** show $\underline{\leftrightarrow} \neq \underline{\leftrightarrow}_\omega$ via games! $\underline{\leftrightarrow}_\omega$ is not always the greatest fixed point for $f_{\underline{\leftrightarrow}}(Z)$.

The discovery of bisimulation

- In modal logic: van Benthem (1976) (based on the work of Segerberg (1971), de Jongh and Troelstra (1966) + the insight of a relational definition)
- In Computer Science: Park (1981) (based on the work of Milner (1980) + the insight of the greatest fixed point)
- In (non-well-founded) Set theory: Forti and Honsell (1981) Hinnion, and it was made popular by Aczel (1988)

Non-wellfounded set theory

Foundation Axiom (FA) implies that there is no infinite "descending" sequence of sets:

$$\cdots \in x_3 \in x_2 \in x_1 \in x_0$$

Two sets are equivalent iff they contain the same elements.

$$A = \{B\}, B = \{A\}, C = \{C, A\}$$

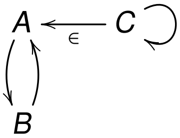
Are A, B and C equivalent?
Basic intuition: if x and y are considered equivalent then *at least* the following should hold:

- for each element x' of x there is an element y' of y such that x' and y' are also considered equivalent,
- for each element y' of y there is an element y' of y such that x' and y' are also considered equivalent.

Non-wellfounded set theory to modal logic

$$A = \{B\}, B = \{A\}, C = \{C, A\}$$

The sets and the membership relation can be viewed as a graph:



The total relation $Z = \{A, B, C\} \times \{A, B, C\}$ is indeed a bisimulation relation, thus A, B, C are bisimilar to each other. The sets $A = \{B\}, B = \{A\}, C = \{C, A\}$ can be viewed equivalent.

Further readings

- D. Sangiorgi, On the origins of Bisimulation and Coinduction. *ACM Transactions on Programming Languages and Systems*, Vol. 31, No. 4, 2009.
- D. Sangiorgi, Introduction to Bisimulation and Coinduction. Cambridge University Press, 2011

Circularity in dictionaries

Merriam-Webster dictionary (2013):

- Hill (1): “a usually rounded natural elevation of land **lower than a mountain**”
- Mountain (1a): “a landmass that projects conspicuously above its surroundings and is **higher than a hill**”
- Oak (1a): “any of a genus (Quercus) of trees or shrubs of the beech family that produce **acorns**”
- Acorn: “the nut of the **oak tree**”

We can still understand those words due to their connections with other words.