Advanced Modal Logic III

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Advanced Modal Logic (2020 Spring)

When can two pointed models be seen as the same?

What do you think? It depends on the measurement...

- Structural equivalences ($\approx_S$), e.g., isomorphism.
- Logical equivalences ($\equiv_{\mathbf{L}}$) (two models cannot be told apart by any formula of a certain logical language **L**). More precisely, for all $\phi \in \mathbf{L}$, $\mathcal{M}, w \vDash \phi \iff \mathcal{N}, v \vDash \phi$.
- Matching the two: logical characterization of the structural equivalence: $\mathcal{M}, w \equiv_{\mathbf{L}} \mathcal{N}, v \iff \mathcal{M}, w \approx_S \mathcal{N}, v$.
  - to measure the *distinguishing power* of a logic.
  - to see what properties are preserved by the structural equivalence.

If $\mathcal{M}, w \approx_S \mathcal{N}, v$ implies $\mathcal{M}, w \equiv_{\mathbf{L}} \mathcal{N}, v$ then we say **L** is *invariant* under $\approx_S$ (or $\approx_S$ is an *invariance* for **L**).

Equivalence notions: examples

For simplicity, we ignore the valuation first. Consider the models for a language with finitely many unary modalities: $\Box_a, \Box_b, \Box_c \ldots$

Trace equivalence

Let $T(\mathcal{M}, w) = \{s \mid s = a_0 ... a_k$ and there is a path $w \xrightarrow{a_0} w_0 \xrightarrow{a_1} \ldots \xrightarrow{a_k} w_k$ in $\mathcal{M}\}$. $\mathcal{M}, w \approx_{trace} \mathcal{N}, v \iff T(\mathcal{M}, w) = T(\mathcal{N}, v)$.
Given $\mathcal{M}$, $w \xrightarrow{a} v$ is simply the abbreviation of $(w, v) \in R_a^{\mathcal{M}}$.

$T(\mathcal{M}, w) = \{a, b\}$, $T(\mathcal{N}, v) = \{a, b\}$.
Q: Which syntactic fragment of modal logic can characterize this equivalence notion? (Hint: we only need formulas like $\Diamond_{a_1} \ldots \Diamond_{a_n} \top$)

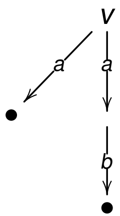
Equivalence notions via behaviours

Complete trace equivalence

$CT(\mathcal{M}, w) = \{s \mid s = a_0 ... a_k$ and there is a path $w \xrightarrow{a_0} w_0 \xrightarrow{a_1} \dots \xrightarrow{a_k} w_k$ in $\mathcal{M}$ such that w_k does not have successors$\}$

$\mathcal{M}, w \approx_{ctrace} \mathcal{N}, v \iff (T(\mathcal{M}, w) = T(\mathcal{N}, v)$ and $CT(\mathcal{M}, w) = CT(\mathcal{N}, v))$.

$\approx_{trace}$ but $\not\approx_{ctrace}$



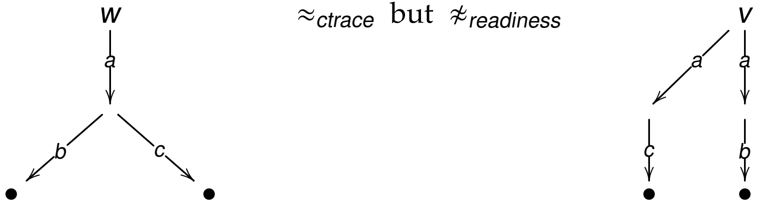
$CT(\mathcal{M}, w) = \{ab\}, CT(\mathcal{N}, v) = \{a, ab\}$

Equivalence notions

Readiness equivalence

Let $Rd(\mathcal{M}, w) = \{(s, A) \mid s = a_0 ... a_k,$ and there is a path $w \xrightarrow{a_0} w_0 \xrightarrow{a_1} \ldots \xrightarrow{a_k} w_k$ in $\mathcal{M}$ and $A = \{a \mid w_k \xrightarrow{a} v\}\}$.
$\mathcal{M}, w \approx_{readiness} \mathcal{N}, v \iff Rd(\mathcal{M}, w) = Rd(\mathcal{N}, v)$.

$\approx_{ctrace}$ but $\not\approx_{readiness}$



$Rd(\mathcal{M}, w) = \{(a, \{b, c\}), (ab, \emptyset), (ac, \emptyset)\}$
$Rd(\mathcal{N}, v) = \{(a, \{c\}), (a, \{b\}), (ac, \emptyset), (ab, \emptyset)\}$

Equivalence notions

Readiness equivalence

Let $Rd(\mathcal{M}, w) = \{(s, A) \mid s = a_0 ... a_k,$ and there is a path $w \xrightarrow{a_0} w_0 \xrightarrow{a_1} \ldots \xrightarrow{a_k} w_k$ in $\mathcal{M}$ and $A = \{a \mid w_k \xrightarrow{a} v$ for some $v\}\}$.
$\mathcal{M}, w \approx_{readiness} \mathcal{N}, v \iff Rd(\mathcal{M}, w) = Rd(\mathcal{N}, v)$.

A logical characterization via language $\mathbf{L}^{Rd}$

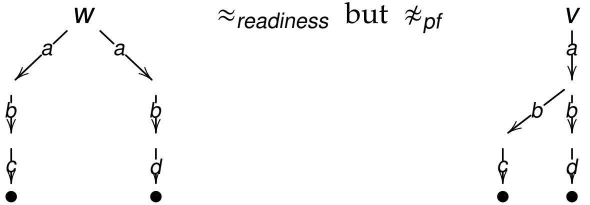
$$\phi \quad ::= \quad \bigwedge_{a \in A} \Diamond_a \top \wedge \bigwedge_{a \notin A} \neg \Diamond_a \top \mid \Diamond_a \phi$$

where $\Diamond_a := \neg \Box_a \neg$ and $\Box_a \in \mathbf{O}$ (finite), $A \subseteq \{a \mid \Box_a \in \mathbf{O}\}$. Let $\Diamond_A \top$ be the abbreviation for $\bigwedge_{a \in A} \Diamond_a \top \wedge \bigwedge_{a \notin A} \neg \Diamond_a \top$. How to prove the following claim?
For all $\mathcal{M}, w$, $\mathcal{N}, v$: $\mathcal{M}, w \approx_{readiness} \mathcal{N}, v \iff \mathcal{M}, w \equiv_{\mathbf{L}^{Rd}} \mathcal{N}, v$
(From left to right: be careful about the inductive proof!)

Equivalence notions

Possible future equivalence

Let $P(\mathcal{M}, w) = \{(s, X) \mid s = a_0 ... a_k$ and there is a path $w \xrightarrow{a_0} w_0 \xrightarrow{a_1} \dots \xrightarrow{a_k} w_k$ in $\mathcal{M}$ and $X = T(\mathcal{M}, w_k)\}$.
$\mathcal{M}, w \approx_{PF} \mathcal{N}, v \iff P(\mathcal{M}, w) = P(\mathcal{N}, v)$. Q: can you characterize $\approx_{PF}$ by a fragment of modal logic?



$\approx_{readiness}$ but $\not\approx_{pf}$

$P(\mathcal{M}, w) = \{(a, \{b, bc\}), (a, \{b, bd\}), (ab, \{c\}), (ab, \{d\})(abc, \emptyset), (abd, \emptyset)\}$

$P(\mathcal{N}, v) = \{(a, \{b, bc, bd\}), (ab, \{c\}), (ab, \{d\})(abc, \emptyset), (abd, \emptyset)\}$

Equivalence notions

The above notions can be lifted to incorporate valuations.
Clearly $\approx_{PF} \subset \approx_{ready} \subset \approx_{ctrace} \subset \approx_{trace}$.
We say $\approx$ is *finer* than $\approx'$ (or $\approx'$ is *coarser* than $\approx$) if $\approx \subseteq \approx'$.
We need fragments of modal logic which have stronger distinguishing power to characterize finer structural equivalences than the coarser ones.
There are (finite) models that are PF-equivalent but still can be distinguished by a modal formula of the basic similarity type.
Then the question is: what is the structure equivalence notion corresponding to modal equivalence (for the full modal language $\mathtt{ML}$)?

A generalized trace equivalence

$\approx_{PF} \subset \approx_{readiness} \subset \approx_{ctrace} \subset \approx_{trace}$.

Definition (n–nested trace equivalence $\approx_n$ without valuations)

1. for any $\mathcal{M}, w, \mathcal{N}, v$: $\mathcal{M}, w \approx_0 \mathcal{N}, v$
2. $\mathcal{M}, w \approx_{n+1} \mathcal{N}, v$ if:
   - for any *trace* s, if $w \xrightarrow{s} w'$ in $\mathcal{M}$ then there is a v' in $\mathcal{N}$ such that $v \xrightarrow{s} v'$ and $\mathcal{M}, w' \approx_n \mathcal{N}, v'$
   - for any *trace* s, if $v \xrightarrow{s} v'$ in $\mathcal{N}$ then there is a w' in $\mathcal{M}$ such that $w \xrightarrow{s} w'$ and $\mathcal{M}, w' \approx_n \mathcal{N}, v'$

$\approx_1 = \approx_{trace}$ and $\approx_2 = \approx_{PF}$.

For any n, we can find some models $\mathcal{M}, w$ and $\mathcal{N}, v$ such that $\mathcal{M}, w \approx_n \mathcal{N}, v$ but $\mathcal{M}, w \not\equiv_{\mathtt{ML}} \mathcal{N}, v$ (even when we do not use any proposition letters).

A generalized trace equivalence

Language $\mathbf{L}_n$

$$\begin{aligned} \mathbf{L}_0 : \phi &::= \top \mid \neg\top \\ \mathbf{L}_{n+1} : \phi &::= \top \mid \psi \mid \neg\phi \mid (\phi \wedge \phi) \mid \Diamond_s \psi \end{aligned}$$

where $\psi \in \mathbf{L}_n$, s is a trace and $\Diamond_{a_0 \dots a_k}\psi$ is the abbreviation for $\Diamond_{a_0} \dots \Diamond_{a_k}\psi$.

A logical characterization for $\approx_n$ (over finite models)

For any n, any $\mathcal{M}, w$, $\mathcal{N}, v$: $\mathcal{M}, w \approx_n \mathcal{N}, v \iff \mathcal{M}, w \equiv_{\mathbf{L}_n} \mathcal{N}, v$

Useful observations: $\approx_{n+1} \Longrightarrow \approx_n$.
Note that without the restriction on models, the right-to-left direction does not hold in general.

Towards bisimulation (in mathematics): morphisms

We consider a modal language with unary modalities $\Box_a, \Box_b, \Box_c \ldots$.

Morphisms are structure-preserving mappings.

Definition (Homomorphism (for unary similarity types))

A *homomorphism* from $\mathcal{M}$ to $\mathcal{N}$ is a function $f : W_{\mathcal{M}} \to W_{\mathcal{N}}$ such that:

- For any proposition letter $p \in \mathbf{P}$:
  $p \in V(w) \implies p \in V(f(w))$;
- For all a: if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then $f(w) \xrightarrow{a} f(w')$ in $\mathcal{N}$.

Strong homomorphism and isomorphism

Definition (Strong Homomorphism (for unary similarity types))

A strong homomorphism from $\mathcal{M}$ to $\mathcal{N}$ is a function $f : W_{\mathcal{M}} \to W_{\mathcal{N}}$ such that:

- For any proposition letter $p \in \mathbf{P}$: $p \in V(w) \iff p \in V(f(w))$;
- For all a: $w \xrightarrow{a} w'$ in $\mathcal{M} \iff f(w) \xrightarrow{a} f(w')$ in $\mathcal{N}$.

An injective strong homomorphism is called an *embedding*.
A bijective strong homomorphism is called an *isomorphism*. We say two pointed models $\mathcal{M}, w$ and $\mathcal{N}, v$ are *isomorphic* to each other ($\mathcal{M}, w \cong \mathcal{N}, v$) if there is an isomorphism f between $\mathcal{M}$ and $\mathcal{N}$ such that $f(w) = v$.

Strong homomorphism and isomorphism

Isomorphism preserves modal formulas: $\cong \subseteq \equiv_{\mathtt{ML}}$, but it is too fine to match $\equiv_{\mathtt{ML}}$. Strong homomorphism does not preserve modal formulas but surjective strong homomorphism does. However, it is still too strong, e.g., there is no surjective strong homomorphism from $\mathcal{M}$ to $\mathcal{N}$ linking w and v :

from homomorphism to bounded morphism

Idea: change the second *iff* condition to *forward and backward* conditions.

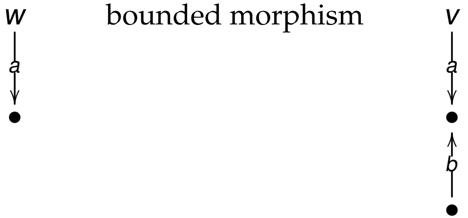
Definition (Bounded morphism (p-morphism))

A bounded morphism from $\mathcal{M}$ to $\mathcal{N}$ is a function $f : W_{\mathcal{M}} \to W_{\mathcal{N}}$ such that:

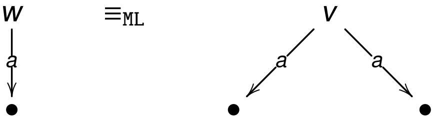
- For any proposition letter $p \in \mathbf{P}$:
  $p \in V(w) \iff p \in V(f(w))$;
- For all a: if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then $f(w) \xrightarrow{a} f(w')$ in $\mathcal{N}$.
- For all a: if $f(w) \xrightarrow{a} v'$ in $\mathcal{N}$ then there is a w' in $\mathcal{M}$ such that $f(w') = v'$ and $w \xrightarrow{a} w'$ in $\mathcal{M}$.

Bounded morphism

Seems OK:



But



Bisimulation

From a function to a relation!

Definition (Bisimulation)

A *non-empty* binary relation R between the domains of two models $\mathcal{M}$ and $\mathcal{N}$ is called a *bisimulation* iff whenever $(w, v) \in R$ the following conditions are satisfied:

- **Invariance** For any $p \in \mathbf{P} : p \in V_{\mathcal{M}}(w) \iff p \in V_{\mathcal{N}}(v)$.
- **Zig** if $w \xrightarrow{a} w'$ in $\mathcal{M}$ then there exists a v' in $\mathcal{N}$ such that $v \xrightarrow{a} v'$ and $w' R v'$.
- **Zag** if $v \xrightarrow{a} v'$ in $\mathcal{N}$ then there exists a w' in $\mathcal{M}$ such that $w \xrightarrow{a} w'$ and $w' R v'$.

$(\mathcal{M}, w)$ and $(\mathcal{N}, v)$ are said to be *bisimilar* $(\mathcal{M}, w \leftrightarrows \mathcal{N}, v)$ if there is a bisimulation R between them such that $(w, v) \in R$.