Beyond knowing that

Yanjing Wang

Department of Philosophy, Peking University

Dec. 22nd, 2014

Beyond "knowing that": motivation

Knowledge is not only expressed in terms of "knowing that":

- I *know whether* the claim is true.
- I *know what* your password is.
- I *know how* to go to Tsinghua.
- I *know who* proved this theorem.
- ...

Linguistically: "know" takes embedded questions but "believe" does not; ambiguity in concealed questions.

Philosophically: are they reducible to "knowing that"?

Logically: how to reason about those forms of knowledge?

Computationally: how to efficiently represent and do inference about those knowledge expressions?

Beyond knowing that: research agenda

In fact, "knowing who" was briefly discussed already by Hinttikka (1962) in terms of first-order modal logic: $\exists x K_i(Beihai = x)$.

Our agenda:

- Keep the language *neat* and take know-constructions as they are, e.g., pack $\exists x K_i(Beihai = x)$ into *Kwho* Beihai.
- Give an intuitive semantics according to some linguistic theory.
- Axiomatize the logics with (combinations of) those operators.
- Dynamify those logic with knowledge updates.
- Automate the inferences.
- Come back to philosophy and linguistics with new insights.

Beyond knowing that: difficulties and some results

New operators behave quite differently from the standard modal operator and are usually disguised FO-modal fellows, which causes difficulties for axiomatization and decidable machinery.

Some of our results:

- Knowing whether (non-contingency): axiomatizations and completeness proofs for its logic over various frame classes [Fan, Wang & van Ditmarsch: AiML14, RSL14 to appear]
- Knowing what: axiomatization and decidability for conditionally knowing what logic over FO epistemic models [Wang & Fan: IJCAI13, AiML14][Xiong 14]
- Knowing how: philosophical discussion [Lau & Wang]; alternative non-possible-world semantics [Wang ICLA14]

Knowing whether operator Kw_i [HHS96, MR66, vdHL03]

KwL is defined as follows:

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid Kw_i\phi$$

where $p \in \mathbf{P}$ and $i \in \mathbf{I}$.
$Kw_i\phi$ says 'agent i knows whether ϕ is true'.

$$\begin{array}{|lcl|} \mathcal{M}, s \vDash Kw_i\phi & \Leftrightarrow & \text{for all } t_1, t_2 \text{ such that } s \to_i t_1, s \to_i t_2 : \\ & & (\mathcal{M}, t_1 \vDash \phi \Leftrightarrow \mathcal{M}, t_2 \vDash \phi) \end{array}$$

In a non-epistemic setting: ϕ is non-contingent, one has opinion about ϕ...
Let **PALKw** be **KwL** extended with the announcement operator.

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid Kw_i\phi \mid \langle\phi\rangle\phi$$

Clearly, **KwL** is no more expressive than **EL** since we can define a translation t : **KwL** $\rightarrow$ **EL** such that:

$$t(Kw_i\phi) \quad = \quad K_i t(\phi) \vee K_i \neg t(\phi)$$

What about the other way around? It depends on the class of models if we do not restrict ourselves to S5 models (models with equivalence relations).

Proposition (Cresswell 88)

KwL *is equally expressive as* **EL** *on the class of reflexive models but it is strictly less expressive than* **EL** *on the class of arbitrary models.*

But how to characterize the expressive power of **KwL** over arbitrary models?
How to give a complete axiomatization? (**KwL** is not normal)

A crucial observation

K_i is *almost definable* by Kw_i:

Proposition

For any ϕ, ψ, $\vDash \neg Kw_i\psi \to (K_i\phi \leftrightarrow Kw_i\phi \wedge Kw_i(\psi \to \phi))$.

This inspires us to come up with a structural equivalence notion of *Kw_i-bisimulation* which plays the role of bisimulation in **EL** (preserving the **KwL** formulas).

Definition (Δ-Bisimulation)

Let $\mathcal{M} = \langle S, R, V \rangle$ be a model. A binary relation Z over S is a Δ-*bisimulation* on $\mathcal{M}$, if Z is non-empty and whenever sZs':

- (Invariance) s and s' satisfy the same propositional variables;
- (Zig) if there are two successors t_1, t_2 of s such that $(t_1, t_2) \notin Z$ and sRt, then there is a t' such that $s'Rt'$ and tZt';
- (Zag) if there are two successors t'_1, t'_2 of s' such that $(t'_1, t'_2) \notin Z$ and $s'Rt'$, then there is a t such that sRt and tZt'.

Theorem ([FWvD14])

Over the class of arbitrary models, an **EL***-formula is equivalent to an* **KwL***-formula iff it is invariant under Kw-bisimulation.*

Also by using the *Kw*-bisimulation, we can also show that:

Theorem ([FWvD14])

The frame properties of seriality, reflexivity, transitivity, symmetry, and Euclidicity are **not** *definable in* **KwL**.

This shows that it is impossible to use **KwL** formulas to capture frame properties, thus it may be hard to axiomatize **KwL** logics over the usual frame classes.

We proposed the following axioms and rules as system $\mathbb{SPLKW}$:

`TAUT`	all instances of tautologies
`KwCon`	$Kw_i(\chi \to \phi) \wedge Kw_i(\neg\chi \to \phi) \to Kw_i\phi$
`KwDis`	$Kw_i\phi \to Kw_i(\phi \to \psi) \vee Kw_i(\neg\phi \to \chi)$
`Kw`$\leftrightarrow$	$Kw_i\phi \leftrightarrow Kw_i\neg\phi$
`MP`	From ϕ and $\phi \to \psi$ infer ψ
`NECKw`	From ϕ infer $Kw_i\phi$
`REKw`	From $\phi \leftrightarrow \psi$ infer $Kw_i\phi \leftrightarrow Kw_i\psi$

Theorem ([FWvD13])

$\mathbb{SPLKW}$ *is sound and complete w.r.t.* **KwL** *over the class of arbitrary frames.*

The proof is based on the following cannonical model construction, inspired by the "almost definability":

Definition (Canonical model)

Define $\mathcal{M}^c = \langle S^c, R^c, V^c \rangle$ as follows:

- $S^c = \{s \mid s$ is a maximal consistent set of $\mathbb{SPLKW}\}$
- For all $s, t \in S^c$, $sR^c_i t$ iff there exists χ such that:
  - $\neg Kw_i \chi \in s$, and
  - for all ϕ, $Kw_i \phi \wedge Kw_i(\chi \to \phi) \in s$ implies $\phi \in t$.
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

The truth lemma relies on `KwCon` and `KwDis`.
To construct canonical models of **KwL** over other class of frames, we need to revise the canonical model case by case.

Definition (Extensions of $\mathbb{SPLKW}$)

Notation	Axiom Schemas	Systems
KwT	$Kw_i\phi \land Kw_i(\phi \to \psi) \land \phi \to Kw_i\psi$	$\mathbb{PLKWT} = \mathbb{SPLKW} + \texttt{KwT}$
Kw4	$Kw_i\phi \to Kw_i(Kw_i\phi \lor \psi)$	$\mathbb{PLKW}4 = \mathbb{SPLKW} + \texttt{Kw4}$
KwB	$\phi \to Kw_i((Kw_i\phi \land Kw_i(\phi \to \psi) \land \neg Kw_i\psi) \to \chi)$	$\mathbb{PLKWS}4 = \mathbb{SPLKW} + \texttt{KwB}$
Kw5	$\neg Kw_i\phi \to Kw_i(\neg Kw_i\phi \lor \psi)$	$\mathbb{PLKW}5 = \mathbb{SPLKW} + \texttt{Kw5}$
wKw4	$Kw_i\phi \to Kw_iKw_i\phi$	$\mathbb{PLKWS}4 = \mathbb{SPLKW} + \texttt{KwT} +$
wKw5	$\neg Kw_i\phi \to Kw_i\neg Kw_i\phi$	$\mathbb{PLKWS}5 = \mathbb{SPLKW} + \texttt{KwT} +$

Theorem ([FWvD13, FWvD14])

(1)The above systems are sound and complete w.r.t. **KwL** *over the corresponding frame classes. (2)The above systems extended with the reduction axioms and the following extra one are sound and complete w.r.t.* **PALKw** *over the corresponding frame classes:*

$$[\phi]Kw_i\psi \leftrightarrow (\phi \to (Kw_i[\phi]\psi \lor Kw_i[\phi]\neg\psi))$$

Knowing what operator Kv_i proposed by [Pla89]

ELKv is defined as (where $c \in C$):

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid K_i\phi \mid Kv_i c$$

ELKv is interpreted on FO-epistemic models with constant domain $\mathcal{M} = \langle S, D, \{\sim_i \mid i \in \mathbf{I}\}, V, V_C \rangle$ where D is a *constant* domain, V_C assigns to each (non-rigid) $c \in C$ a $d \in D$ on each $s \in S$:

$$\mathcal{M}, s \vDash Kv_i c \iff \text{for any } t_1, t_2: \text{ if } s \sim_i t_1, s \sim_i t_2, \text{ then } V_C(c, t_1) = V_C(c, t_2).$$

ELKv can express "*i* knows that *j* knows the password but *i* doesn't know what exactly it is" by $K_i Kv_j c \wedge \neg Kv_i c$.

The interaction between the two operators is crucial: it cannot be treated as $K_i K_j p \wedge \neg K_i p$.

Knowing what operator Kv_i proposed by [Pla89]

To handle the *Sum and Product* puzzle, Plaza extended **ELKv** with announcement operator (call it **PALKv**):

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid K_i\phi \mid Kv_i c \mid \langle\phi\rangle\phi$$

[Pla89] proposed a system $\mathbb{PALKV}_p$ for **PALKv** on top of $\mathbb{S}5$.

Theorem ([WF13])

$\langle p\rangle Kv_i c \wedge \langle q\rangle Kv_i c \to \langle p \vee q\rangle Kv_i c$ is not derivable in $\mathbb{PALKV}_p$, thus $\mathbb{PALKV}_p$ is not complete w.r.t. $\vDash$ on FO-epistemic models.

Conditionally knowing what

Axiomatizing **PALKv** is indeed hard. We propose a conditional generalization of Kv_i operator:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid Kv_i(\phi, c)$$

where $Kv_i(\phi, c)$ says "agent i knows what c is *given* ϕ", e.g., I know my password for this website if it is 4-digit. More precisely, agent i *would know* what c is if he is informed that ϕ.

$$\mathcal{M}, s \vDash Kv_i(\phi, c) \quad \Leftrightarrow \quad \text{for any } t_1, t_2 \in S \text{ such that } s \sim_i t_1 \text{ and } s \sim_i t_2 : \mathcal{M}, t_1 \vDash \phi \& \mathcal{M}, t_2 \vDash \phi \text{ implies } V_C(c, t_1) = V_C(c, t_2)$$

Let $\mathbf{PALKv}^r$ be:

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid K_i\phi \mid Kv_i(\phi, c) \mid \langle\phi\rangle\phi$$

PALKv^r looks more expressive than **PALKv** but in fact they are equally expressive.

Theorem ([WF13])

The comparison of the expressive power of those logics are summarized in the following (transitive) diagram:

$$\begin{array}{ccc} \mathbf{ELKv}^r & \longleftrightarrow & \mathbf{PALKv}^r \\ \uparrow & & \updownarrow \\ \mathbf{ELKv} & \longrightarrow & \mathbf{PALKv} \end{array}$$

where **ELKv** and **ELKv**^{r} are the announcement-free fragments of **PALKv** and **PALKv**^{r}.

We can simply forget about Plaza's **PALKv** and use **ELKv**^{r}!

System $\mathbb{ELKV}^r$

Axiom Schemas	
TAUT	all the instances of tautologies
DISTK	$K_i(p \to q) \to (K_i p \to K_i q)$
T	$K_i p \to p$
4	$K_i p \to K_i K_i p$
5	$\neg K_i p \to K_i \neg K_i p$
DISTKv^r	$K_i(p \to q) \to (Kv_i(q, c) \to Kv_i(p, c))$
Kv^{r}4	$Kv_i(p, c) \to K_i Kv_i(p, c)$
Kv$^r\bot$	$Kv_i(\bot, c)$
Kv$^r\vee$	$\hat{K}_i(p \wedge q) \wedge Kv_i(p, c) \wedge Kv_i(q, c) \to Kv_i(p \vee q, c)$

Rules	
MP	$\dfrac{p, p \to q}{q}$
NECK	$\dfrac{\phi}{K_i\phi}$
SUB	$\dfrac{\phi}{\phi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\phi \leftrightarrow \phi[\psi/\chi]}$

$Kv_i(\phi, c)$ can be viewed as $\exists x K_i(\phi \to c = x)$ where x is a *rigid* variable and c is a *non-rigid* one.

A Kv_i operator packages a quantifier, a modality, an implication and an equality together: a blessing and a curse.

To build a suitable canonical FO-epistemic model with a constant domain, we need to saturate each maximal consistent set with:

- counterparts of atomic formulas such as $c = x$
- counterparts of $K_i(\phi \to c = x)$

By using axioms in the modal language, we need to make sure these extra bits are consistent with the maximal consistent sets and canonical relations.

Definition

Let MCS be the set of maximal consistent sets w.r.t. $\mathbb{ELKV}^r$, and let $\mathbb{N}$ be the set of natural numbers. The canonical model $\mathcal{M}$ of $\mathbb{ELKV}^r$ is a tuple $\langle S, \mathbb{N}, \{\sim_i \mid i \in \mathbf{I}\}, V, V_C\rangle$ where:

- S consists of all the triples $\langle \Gamma, f, g\rangle \in MCS \times \mathbb{N}^C \times (\mathbb{N} \cup \{\star\})^{\mathbf{I} \times \mathbf{ELKv}^r \times C}$ that satisfy the following three conditions for any $i \in \mathbf{I}$, any $\psi, \phi \in \mathbf{ELKv}^r$, and any $d \in C$:
  - (i) $g(i, \psi, d) = \star$ iff $Kv_i(\psi, d) \wedge \hat{K}_i\psi \notin \Gamma$,
  - (ii) If $g(i, \phi, d) \neq \star$ and $g(i, \psi, d) \neq \star$ then: $g(i, \phi, d) = g(i, \psi, d)$ iff $Kv_i(\phi \vee \psi, d) \in \Gamma$
  - (iii) $\psi \wedge Kv_i(\psi, d) \in \Gamma$ implies $f(d) = g(i, \psi, d)$.
- $s \sim_i t$ iff $\{\phi \mid K_i\phi \in s\} \subseteq t$ and $g_s(i) = g_t(i)$
- $V_C(d, s) = f_s(d)$

Lemma

Each maximal consistent set can be properly saturated with those counterparts.

Lemma

Each saturated MCS including $\Diamond\phi$ has a saturated ϕ-successor.

Lemma

Each saturated MCS including $\neg Kv_i(\phi, c)$ has two saturated ϕ-successors which disagree about the value of c.

Axiom $\texttt{Kv}^r\vee$: $\hat{K}_i(p \wedge q) \wedge Kv_i(p, c) \wedge Kv_i(q, c) \rightarrow Kv_i(p \vee q, c)$ plays an extremely important role.

Theorem ([WF13, WF14])

$\mathbb{ELKV}^r$ *is sound and strongly complete for* $\mathbf{ELKv}^r$*.*

Theorem ([Xio14])

$\mathbf{ELKv}^r$ *on epistemic models is decidable.*

Theorem (Ding)

W.r.t. the class of all models: $\mathbb{ELKV}^r$ *without* $\mathtt{T}$*,*$\mathtt{4}$*,*$\mathtt{5}$ *is complete and* $\mathbf{ELKv}^r$ *is PSPACE-complete.*

We can axiomatize multi-agent $\mathbf{PALKv}^r$ by adding the following reduction axiom schemas (call the resulting system $\mathbb{SPALKV}^r$):

$\mathtt{!ATOM}$	$\langle\psi\rangle p \leftrightarrow (\psi \wedge p)$
$\mathtt{!NEG}$	$\langle\psi\rangle\neg\phi \leftrightarrow (\psi \wedge \neg\langle\psi\rangle\phi)$
$\mathtt{!CON}$	$\langle\psi\rangle(\phi \wedge \chi) \leftrightarrow (\langle\psi\rangle\phi \wedge \langle\psi\rangle\chi)$
$\mathtt{!K}$	$\langle\psi\rangle K_i\phi \leftrightarrow (\psi \wedge K_i(\psi \to \langle\psi\rangle\phi))$
$\mathtt{!Kv}^r$	$\langle\phi\rangle Kv_i(\psi, c) \leftrightarrow (\phi \wedge Kv_i(\langle\phi\rangle\psi, c))$

Conclusions

We study modal logics with on the know-whether operator Kw_i and the conditional know-what operator Kv_i.

The main results:

- We give complete axiomatizations of **KwL** and **PALKw** over various classes of frames.
- We give complete axiomatizations of $\mathbf{ELKv}^r$ and $\mathbf{PALKv}^r$ over S5 frames.
- The axioms involved are quite different from the normal modal logics and the completeness proofs require new techniques.

[FWvD13] Jie Fan, Yanjing Wang, and Hans van Ditmarsch. Knowing whether. under submission, Nov. 2013.

[FWvD14] Jie Fan, Yanjing Wang, and Hans van Ditmarsch. Almost neccessary. In Proceedings of AiML14, April 2014.

[HHS96] S. Hart, A. Heifetz, and D. Samet. Knowing whether, knowing that, and the cardinality of state spaces. *Journal of Economic Theory*, 70(1):249–256, 1996.

[MR66] H. Montgomery and R. Routley. Contingency and non-contingency bases for normal modal logics. *Logique et Analyse*, 9:318–328, 1966.

[Pla89] J. A. Plaza. Logics of public communications. In M. L. Emrich, M. S. Pfeifer, M. Hadzikadic, and Z. W. Ras, editors, *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems*, pages 201–216, 1989.

[vdHL03] W. van der Hoek and A. Lomuscio. Ignore at your peril - towards a logic for ignorance. In *Proc. of 2nd AAMAS*, pages 1148–1149. ACM, 2003.

[WF13] Yanjing Wang and Jie Fan. Knowing that, knowing what, and public communication: Public announcement logic with Kv operators. In *Proceedings of IJCAI*, pages 1139–1146, 2013.

[WF14] Yanjing Wang and Jie Fan. Conditionally knowing what. in proceedings of AiML14, April 2014.

[Xio14] Shihao Xiong. Decidability of $ELKv^r$. Bachelaor thesis, May 2014.